Calculus 2 for Honours Mathematics

MATH 138

Winter 2019 (1191)

LaTeXer: *Cameron Roopnarine*

Instructor: *Jordan Hamilton*

May 19, 2021

Contents

Chapter 1

Integration

1.2 Riemann Sums and the Definite Integral

To begin with, our goal is to develop methods for determining the area under a curve.

We know we can approximate the area using rectangles (or other geometric shapes), but we want the *exact* area. For this, we will need *Riemann sums*.

DEFINITION 1.2.1: Partition

A **partition**, P, for the interval $[a, b]$ is a finite sequence of increasing numbers of the form

$$a = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = b$$

This partition subdivides the interval $[a, b]$ into n subintervals:

$$[t_0, t_1], \dots, [t_{n-1}, t_n]$$

REMARK 1.2.2

These subintervals may *not* all have the same length.

DEFINITION 1.2.3: Length

Denote the **length** of the i^{th} subinterval, $[t_{i-1}, t_i]$, by Δt_i; that is, $\Delta t_i = t_i - t_{i-1}$.

DEFINITION 1.2.4: Norm

The **norm** of a partition is the length of the widest subinterval:

$$\|P\| = \max(\Delta t_1, \dots, \Delta t_n)$$

DEFINITION 1.2.5: Riemann sum

Given a bounded function f on $[a, b]$, a partition P of $[a, b]$, and a set $\{c_1, \dots, c_n\}$, where $c_i \in [t_{i-1}, t_i]$, then a **Riemann sum** for f with respect to P is

$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Again, we want the *exact* area, and for that we will need to use infinitely many points!

But we do need to make sure that the norm of our partitions is getting smaller, and that the area we get doesn't depend on the choice of Riemann sum.

DEFINITION 1.2.6: Integrable, Integral of f

We say that f is **integrable** on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated to the P_n's, we have $\lim_{n\to\infty} S_n = I$.
In this case, we call I the **integral of** f over $[a, b]$ and denote it by

$$\int_a^b f(x)\,dx$$

where a, b are the bounds of integration, $f(x)$ is the integrand, x is the variable of integration. The complete object is called a definite integral.
It represents the exact (signed) area under f.

REMARK 1.2.7

The variable of integration is a *dummy variable* since we can change it into whatever we want, and it won't change the value of the integral; that is,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\cdot)\,d\cdot$$

This looks *horrible* to compute in practice (and it is). The good news is if f is continuous, it's not so bad! (still bad though)

THEOREM 1.2.8: Integrability Theorem for Continuous Functions

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$.

Proof of 1.2.8

Beyond the scope of this course.

This is fantastic! This means that we can *choose* any collection of Riemann sums we want when computing the integral of a continuous function!

Let's examine a "nice" choice: one where the partition is regular and where we just pick the c_i's to be the right-hand endpoints!

DEFINITION 1.2.9: Regular n-partition

For the interval $[a, b]$, the **regular n-partition** where all n subintervals have the same length; that is,

$$\Delta t = \frac{b-a}{n} \quad \text{and} \quad t_i = t_0 + i\Delta t$$

DEFINITION 1.2.10: Regular right-hand Riemann sum

Using this, we define the **regular right-hand Riemann sum** by taking $c_i = t_i$ for all i:

$$S_n = \sum_{i=1}^{n} f(t_i)\Delta t = \sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)$$

REMARK 1.2.11

We can also define the regular left-hand Riemann sum.

Now, we can write a nicer formula for integrating continuous functions!

If f is continuous, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)}$$

EXAMPLE 1.2.12

Evaluate $\displaystyle\int_0^4 x + x^3\,dx$.

Solution. Since $f(x) = x + x^3$ is continuous, we can use the above formula.
In our case: $\dfrac{b-a}{n} = \dfrac{4}{n}$, and $t_i = 0 + \dfrac{4i}{n} = \dfrac{4i}{n}$.
So, $f(t_i) = \dfrac{4i}{n} + \dfrac{64i^3}{n^3}$. Then, we get:

$$\begin{aligned}
\int_0^4 x + x^3\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} + \frac{64i^3}{n^3}\right)\left(\frac{4}{n}\right) && (1.1)\\
&= \lim_{n\to\infty} \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{256}{n^4}\sum_{i=1}^{n} i^3 && (1.2)\\
&= \lim_{n\to\infty} \frac{16}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{256}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] && (1.3)\\
&= \lim_{n\to\infty} \frac{8n+8}{n} + 64\left(\frac{n^2+2n+1}{n^2}\right) && (1.4)\\
&= 8 + 64 && (1.5)\\
&= 72 && (1.6)
\end{aligned}$$

where from 1.2 to 1.3 we used both of the following:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

REMARK 1.2.13

The theorem also holds for functions that are bounded and have finitely many discontinuities.

1.3 Properties of the Definite Integral

Since a definite integral is the limit of a sequence, many limit laws also hold!

THEOREM 1.3.1: Properties of Integrals

Assume that f and g are integrable on the interval $[a,b]$. Then:

(1) *For any $c \in \mathbb{R}$, $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.*

(2) $\displaystyle\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(3) *If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.*

(4) *If $0 \leq f(x)$ for all $x \in [a,b]$, then $\displaystyle 0 \leq \int_a^b f(x)\,dx$.*

(5) *If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.*

(6) *The function $|f|$ is integrable on $[a,b]$ and $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.*

Proof of 1.3.1

- (1) and (2) follow from limit laws for sequences.
- (3) implies (4).
- (1), (2), and (4) imply (5).
- (6) follows from the triangle inequality.

We will now prove (3).

Suppose $m \leq f(x) \leq M$ and partition the interval $a = t_0 < \cdots < t_n = b$.

Note that $\displaystyle\sum_{i=1}^n \Delta t = \frac{b-a}{n}(n) = b-a$ Then, since $m \leq f(x) \leq M$, we get

$$m(b-a) = \sum_{i=1}^n m\Delta t \leq \sum_{i=1}^n f(t_i)\Delta t \leq \sum_{i=1}^n M\Delta t = M(b-a)$$

So, taking limits gives

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

DEFINITION 1.3.2: More properties

(I) If $f(a)$ is defined, then $\displaystyle\int_a^a f(x)\,dx = 0$

(II) If f is integrable on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

THEOREM 1.3.3

If f is integrable on an interval I containing a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof of 1.3.3

Beyond the scope of this course.

REMARK 1.3.4

c does *not* need to be between a and b!

Geometric Interpretation of the Integral

So far, we have only examined positive functions, but we should note that $\int_a^b f(x)\,dx$ returns the *signed* area between f and the x-axis. That is, if $f(x) \leq 0$, then $\int_a^b f(x)\,dx \leq 0$ too.

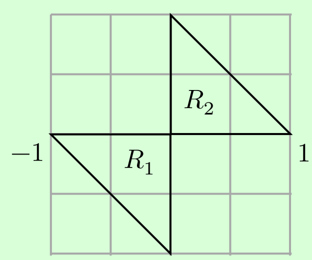
So, in general, $\int_a^b f(x)\,dx$ is the area under f that lies above the x-axis *minus* the area above the graph of f that lies below the x-axis.

EXAMPLE 1.3.5

$$\int_{-1}^{1} x\,dx = R_2 - R_1$$

but $R_2 = R_2$, so

$$\int_{-1}^{1} x\,dx = 0$$



REMARK 1.3.6

If we are lucky, we can use geometric formulas to evaluate integrals (see pg 26–28 in the notes). However, we are almost never this lucky…

1.4 Average Value of a Function

DEFINITION 1.4.1: Average value

If f is continuous on $[a, b]$, the **average value** of f on $[a, b]$ is defined as $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

Geometric Interpretation

Proof of 1.4.2

If f is continuous on $[a, b]$, EVT says there exists $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for $x \in [a, b]$, and $f(c_1) = m$, $f(c_2) = M$ for some $c_1, c_2 \in [a, b]$.
Also, we know

$$\begin{aligned} m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) &\implies m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M \\ &\iff f(c_1) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(c_2) \end{aligned}$$

IVT says there exists c between c_1 and c_2, so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

THEOREM 1.4.2: Average Value Theorem (AVT)

Assume f is continuous on $[a, b]$. There exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

REMARK 1.4.3

Note that this theorem holds even if $b < a$ since

$$\begin{aligned} f(c) &= \frac{1}{a-b}\int_b^a f(x)dx \\ &= \frac{1}{a-b}\left(-\int_a^b f(x)\,dx\right) \\ &= \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

The big problem we face now is that evaluating $\int_a^b f(x)\,dx$ is monstrously difficult for all but the simplest of functions.

IF ONLY THERE WAS A BETTER WAY!

(spoilers: there's a better way! It's the Fundamental Theorem of Calculus!)

1.5 The Fundamental Theorem of Calculus (Part 1)

The FTC is, essentially, a simple derivative rule. But its consequences are very valuable. The reason is that it provides the link between integral calculus and differential calculus!

We start with integral functions: let f be continuous on $[a, b]$.

Define $G(x) = \displaystyle\int_a^x f(t)\,dt$ for $x \in [a, b]$.

What is $G(x)$? It's the function that returns the signed area under f from a to x.

EXAMPLE 1.5.1

Compute the area of $f(x) = x$ on $[0, 5]$.
Solution.

$$G(x) = \int_0^x t\,dt = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x)(x) = \frac{x^2}{2}$$

Wait a minute! $G'(x) = x = f(x)$. Is this always true?!

THEOREM 1.5.2: Fundamental Theorem of Calculus I (FTC I)

If f is continuous on an open interval I containing $x = a$, and if

$$G(x) = \int_a^x f(t)\,dt$$

Then G is differentiable for all $x \in I$ and $G'(x) = f(x)$; that is, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$.

Proof of 1.5.2

Let f be continuous on I, $G(x) = \displaystyle\int_a^x f(t)\,dt$, and fix $x_0 \in I$.
Let $\varepsilon > 0$ be given. Since f is continuous at x_0, there exists a $\delta > 0$ such that if $0 < |c - x_0| < \delta$, then

$$|f(c) - f(x_0)| < \varepsilon$$

Let $0 < |x - x_0| < \delta$. Then,

$$\begin{aligned}\frac{G(x) - G(x_0)}{x - x_0} &= \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} \\ &= \frac{\int_a^{x_0} f(t)\,dt + \int_{x_0}^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} \\ &= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt\end{aligned}$$

The AVT says there exists c between x and x_0 such that

$$f(c) = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$$

Since $0 < |x - x_0| < \delta$, we get $0 < |c - x_0| < \delta$ too, so

$$\left|\frac{G(x) - G(x_0)}{x - x_0} - f(x_0)\right| = |f(c) - f(x_0)| < \varepsilon$$

This says $G'(x_0) = \displaystyle\lim_{x \to x_0} \frac{G(x) - G(x_0)}{x - x_0} = f(x_0)$.

EXAMPLE 1.5.3

Compute $\dfrac{d}{dx}\displaystyle\int_5^x \sin(t^2)\,dt$

Solution. $\dfrac{d}{dx}\displaystyle\int_5^x \sin(t^2)\,dt = \sin(x^2)$ since $f(t) = \sin(t^2)$ is continuous, by FTC I.

EXAMPLE 1.5.4

Compute $\dfrac{d}{dx}\displaystyle\int_5^{x^2} \sin(t^2)\,dt$.

Solution. Let $G(x) = \displaystyle\int_5^x \sin(t^2)\,dt$, then $G(x^2) = \displaystyle\int_5^{x^2} \sin(t^2)\,dt$. So,

$$\frac{d}{dx}\int_5^{x^2} \sin(t^2)\,dt = \frac{d}{dx}\left[G(x^2)\right] = G'(x^2)(2x) = f(x^2)(2x) = \sin(x^4)(2x)$$

We will see a more general formula next week!

1.6 The Fundamental Theorem of Calculus (Part 2)

It seems like integrating is the opposite operation to differentiation, and it is! We can use antiderivatives to evaluate integrals, as we will see. But first, let's quickly recap what we know about antidifferentiation.

DEFINITION 1.6.1: Antiderivative

Given a function f, an **antiderivative** of f is a function F such that $F'(x) = f(x)$.

REMARK 1.6.2

Antiderivatives are not unique! For example, let $f(x) = 2x$, some antiderivatives of $f(x)$ include:

- $F_1(x) = x^2$
- $F_2(x) = x^2 + 4$
- $F_3(x) = x^2 - \pi$

DEFINITION 1.6.3: Indefinite integral

The collection of all antiderivatives of $f(x)$ is denoted by $\displaystyle\int f(x)\,dx$ and

$$\int f(x)\,dx = F(x) + C$$

where $C \in \mathbb{R}$ and F is any antiderivative. This is called the **indefinite integral**.

REMARK 1.6.4

By the Antiderivative Theorem, we know any two antiderivatives of f differ by a constant.

Here are a bunch of antiderivatives:

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$
- $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
- $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$ for $a > 0$

By 1.5.2, we know every continuous function has an antiderivative, but how can we use them to actually evaluate definite integrals? Well…

THEOREM 1.6.5: Fundamental Theorem of Calculus II (FTC II)

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \big[F(x)\big]_a^b$$

Proof of 1.6.5

Let F be any antiderivative of f and define $G(x) = \displaystyle\int_a^x f(t)\,dt$.

By 1.5.2, we know $G'(x) = f(x)$ as well, so by the Antiderivative Theorem, $G(x) = F(x) + C$ for some $C \in \mathbb{R}$.

But then,

$$G(b) - G(a) = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Also,

$$\begin{aligned} \int_a^b f(t)\,dt &= G(b) \\ &= G(b) - G(a) && \text{since } G(a) = 0 \\ &= F(b) - F(a) \end{aligned}$$

Now, we can evaluate definite integrals *without* Riemann sums!

EXAMPLE 1.6.6

(i) Compute $\displaystyle\int_1^3 x^2 + x\,dx$.

Solution.

$$\int_1^3 x^2 + x\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^3 = \left(\frac{3^3}{3} + \frac{3^2}{2}\right) - \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{27}{2} - \frac{5}{6} = \frac{38}{3}$$

(ii) Compute $\displaystyle\int_0^{2\pi} \sin(x)\,dx$.

Solution.

$$\int_0^{2\pi} \sin(x)\,dx = [-\cos(x)]_0^{2\pi} = -\cos(2\pi) + \cos(0) = 1 + 1 = 0$$

This makes sense since the signed area is zero.

(iii) Compute $\displaystyle\int_2^8 \frac{x^2 + 2x + 1}{x}\,dx$.

Solution.

$$\begin{aligned}\int_2^8 \frac{x^2 + 2x + 1}{x}\,dx &= \int_2^8 x + 2 + \frac{1}{x}\,dx \\ &= \left[\frac{x^2}{2} + 2x + \ln|x|\right]_2^8 \\ &= [32 + 16 + \ln(8)] - [2 + 4 + \ln(2)] \\ &= 42 + \ln(8) - \ln(2) \\ &= 42 + \ln(4)\end{aligned}$$

This is fantastic! We are only limited by our ability to find antiderivatives! As we will see, finding antiderivatives is hard in general, but in the next couple of weeks we will learn a few techniques.

But first, let's look at the extended version of FTC I:

COROLLARY 1.6.7: Extended Version of the Fundamental Theorem of Calculus

If f is continuous, and g, h are both differentiable, then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))h'(x) - f(g(x))g'(x)$$

(also called the Leibniz Formula).

Proof of 1.6.7

Let F be an antiderivative of f, then by FTC II:

$$\int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

for each x. So,

$$\begin{aligned}\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] &= \frac{d}{dx}[F(h(x)) - F(g(x))] \\ &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x)\end{aligned}$$

EXAMPLE 1.6.8

Compute $\displaystyle\frac{d}{dx}\int_{5x}^{\ln(x)} \cos(t^2 - 3t)\,dt$.

Solution.

$$\frac{d}{dx}\int_{5x}^{\ln(x)} \cos(t^2 - 3t)\,dt = \cos[\ln(x)^2 - 3\ln(x)]\left(\frac{1}{x}\right) - \cos(25x^2 - 15x)(5)$$

1.7 Change of Variables

The first integration technique we will examine is the reverse chain rule: Change of Variable, also called Substitution.

The rule is: $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$; that is, we "substitute" $u = g(x)$.

Proof of Change of Variable (Sketch)

Let f and g be functions and let h be an antiderivative of f, so $h'(x) = f(x)$.
Let $H(x) = h(g(x))$, so

$$H'(x) = h'(g(x))g'(x) = f(g(x))g'(x)$$

so $h(g(x))$ is an antiderivative of $f(g(x))g'(x)$.
Therefore,

$$\begin{aligned}\int f(g(x))g'(x)dx &= h(g(x)) + C && \text{for some } c \in \mathbb{R} \\ &= h(u) + C && \text{where } u = g(x) \\ &= \int f(u)\,du && \text{where } u = g(x)\end{aligned}$$

So, if $u = g(x)$, then $du = g'(x)\,dx$.

General strategy: let $u = \ldots$, $du = \ldots\ dx$, then solve for dx, substitute in u and dx, try to transform the integral into one in terms of only u.

Good choices for u:

- $u =$ a function whose derivative is present.
- $u =$ base of an ugly power
- $u =$ function inside another function; that is, inside sin / cos / ln, or in the exponent of e.

EXAMPLE 1.7.1

(i) Compute $\displaystyle\int \frac{\ln(x)}{x}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{\ln(x)}{x}\,dx &= \int \frac{u}{x}(x)\,du && u = \ln(x) \iff du = \frac{1}{x}\,dx \\
&= \int u\,du \\
&= \frac{u^2}{2} + C \\
&= \frac{[\ln(x)]^2}{2} + C
\end{aligned}$$

(ii) Compute $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx &= \int \frac{\cos(u)}{u}(2u)\,du && u = \sqrt{x} \iff du = \frac{1}{2\sqrt{x}}\,dx \\
&= 2\int \cos(u)\,du \\
&= 2\sin(u) + C \\
&= 2\sin(\sqrt{x}) + C
\end{aligned}$$

(iii) Compute $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\,dx$. Don't forget to eliminate all the x's!

Solution.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x+1}}\,dx &= \int \frac{(u-1)^2}{\sqrt{u}}\,du && u = x+1 \iff du = dx \\
&= \int \frac{u^2-2u+1}{\sqrt{u}}\,du \\
&= \int u^{3/2} - 2u^{1/2} + u^{-1/2}\,du \\
&= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C
\end{aligned}$$

(iv) Compute $\displaystyle\int \sin^6(x)\cos(x)\,dx$.

Solution.

$$\begin{aligned}
\int \sin^6(x)\cos(x)\,dx &= \int u^6\,du && u = \sin(x) \iff du = \cos(x)\,dx \\
&= \frac{u^7}{7} + C \\
&= \frac{\sin^7(x)}{7} + C
\end{aligned}$$

(v) Compute $\displaystyle\int xe^{5x^2}\,dx$.

Solution.

$$\begin{aligned}\int xe^{5x^2}\,dx &= \int \frac{xe^u}{10x}\,du && u = 5x^2 \iff du = 10x\,dx\\ &= \frac{1}{10}\int e^u\,du\\ &= \frac{e^u}{10} + C\\ &= \frac{e^{5x^2}}{10} + C\end{aligned}$$

Substitution and Definite Integrals

Q: What should we do with the limits of integration when making a substitution?

A: We should change them as well!

THEOREM 1.7.2: Change of Variable

If $g'(x)$ is continuous on $[a, b]$ and $f(x)$ is continuous between $g(a)$ and $g(b)$, then

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Proof of 1.7.2

Let $h(u)$ be an antiderivative of $f(u)$. Then $h(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

By FTC II, $\displaystyle\int_a^b f(g(x))g'(x)\,dx = h(g(b)) - h(g(a))$.

But also, $\displaystyle\int_{g(a)}^{g(b)} f(u)\,du = h(g(b)) - h(g(a))$, so we get $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

EXAMPLE 1.7.3

(i) Compute $\displaystyle\int_0^1 e^x \cos(e^x)\,dx$.

Solution.

$$\begin{aligned}\int_0^1 e^x \cos(e^x)\,dx &= \int_1^e \frac{u\cos(u)}{u}\,du && u = e^x \iff du = e^x\,dx\\ &= \int_1^e \cos(u)\,du\\ &= \big[\sin(u)\big]_1^e\\ &= \sin(e) - \sin(1)\end{aligned}$$

(ii) Compute $\displaystyle\int_0^1 \frac{x^3}{1+x^4}\,dx$.

Solution.

$$\begin{aligned}
\int_0^1 \frac{x^3}{1+x^4}\,dx &= \int_1^2 \frac{x^3}{u\cdot 4x^3}\,du \qquad\qquad u = 1+x^4 \iff du = 4x^3\,dx \\
&= \frac{1}{4}\int_1^2 \frac{1}{u}\,du \\
&= \left[\frac{\ln|u|}{4}\right]_1^2 \\
&= \frac{\ln(2)}{4} - \frac{\ln(1)}{4} \\
&= \frac{\ln(2)}{4}
\end{aligned}$$

REMARK 1.7.4

You can also leave the limits of integration in terms of x as long as you make it clear and don't forget to switch back to x at the end before plugging numbers in!

EXAMPLE 1.7.5: Tricky Change of Variable

$$\begin{aligned}
\int \sec(x)\,dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

We made the substitution $u = \sec(x)+\tan(x) \iff du = \sec(x)\tan(x)+\sec^2(x)\,dx$.

REMARK 1.7.6

The trick used in 1.7.5 only works for $\sec(x)$ and $\csc(x)$, so it's not useful to memorize.

EXERCISE 1.7.7

Compute: $\int \tan(x)\,dx$, $\int \cot(x)\,dx$.

Chapter 2

Techniques of Integration

2.1 Trigonometric Substitution

Sometimes, changing x into a trigonometric function can simplify an integral!

There are three situations where this is useful: say $\alpha \in \mathbb{R}$.

If you see:	Try substituting:	Range for θ
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$\theta \in (-\pi/2, \pi/2)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$\theta \in (-\pi/2, \pi/2)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\theta \in [0, \pi/2] \cup [\pi, 3\pi/2]$

REMARK 2.1.1

- The range for θ is important to ensure that $\sin(\theta)/\tan(\theta)/\sec(\theta)$ are invertible (so we can solve for θ in terms of x, if need be).
- No, you don't need to state the range for θ each time.
- The integrand may need to be simplified before a trigonometric substitution can be made.
- Don't forget to change back to x in an indefinite integral.

EXAMPLE 2.1.2

Compute $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+4}}\,dx &= \int \frac{1}{\sqrt{4\tan^2(\theta)+4}}\left(2\sec^2(\theta)\right)\,d\theta && x = 2\tan(\theta) \iff dx = 2\sec^2(\theta)\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta)+1}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{|\sec(\theta)|}\,d\theta \\
&= \int \sec(\theta)\,d\theta && \text{since } \sec(\theta) > 0 \\
&= \ln|\sec\theta + \tan(\theta)| + C \\
&= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C
\end{aligned}$$

where we substituted $x = 2\tan(\theta) \iff \tan(\theta) = \frac{x}{2} \implies \sec(\theta) = \frac{\sqrt{x^2+4}}{2}$ in the last step (try to visualize a triangle).

REMARK 2.1.3

When using a trigonometric substitution, the absolute values will **always** go away due to the choice of θ's!

EXAMPLE 2.1.4

(i) Compute $\displaystyle\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{\sqrt{9-4x^2}}{x^2}\,dx &= 2\int \frac{\sqrt{9/4-x^2}}{x^2}\,dx \\
&= 2\int \frac{\sqrt{9/4-9/4\sin^2(\theta)}\,(3/2\cos(\theta))}{9/4\sin^2(\theta)}\,d\theta \qquad x = 3/2\sin(\theta) \iff dx = 3/2\cos(\theta)\,d\theta \\
&= 2\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{4}{9}\int \frac{\sqrt{1-\sin^2(\theta)}\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= 2\int \frac{|\cos(\theta)|\cos(\theta)}{\sin^2(\theta)}\,d\theta \\
&= 2\int \frac{\cos^2(\theta)}{\sin^2(\theta)}\,d\theta \qquad \text{since } \cos(\theta) > 0 \\
&= 2\int \cot^2(\theta)\,d\theta \\
&= 2\int \csc^2(\theta) - 1\,d\theta \\
&= 2[-\cot(\theta) - \theta] + C \\
&= 2\left[-\frac{\sqrt{9-4x^2}}{2x} - \arcsin\left(\frac{2x}{3}\right)\right] + C
\end{aligned}$$

Where we substituted $x = \frac{3}{2}\sin(\theta) \iff \frac{2x}{3} = \sin(\theta) \implies \theta = \arcsin\left(\frac{2x}{3}\right)$ and $\cot(\theta) = \frac{\sqrt{9-4x^2}}{2x}$ in the last step.

(ii) Compute $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$.

Solution.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-4}}\,dx &= \int \frac{2\sec(\theta)\tan(\theta)}{4\sec^2(\theta)\sqrt{4\sec^2(\theta)-4}}\,d\theta \qquad x = 2\sec(\theta) \iff dx = 2\sec(\theta)\tan(\theta)\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)\sqrt{\sec^2(\theta)-1}}\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)\tan(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{\tan(\theta)}{\sec(\theta)+\tan(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{\sec(\theta)}\,d\theta \\
&= \frac{1}{4}\int \cos(\theta)\,d\theta \\
&= \frac{\sin(\theta)}{4} + C \\
&= \frac{\sqrt{x^2-4}}{4x} + C
\end{aligned}$$

Where we substituted $x = 2\sec(\theta) \implies \sin(\theta) = \frac{\sqrt{x^2-4}}{x}$ in the last step.

(iii) Compute $\displaystyle\int x\sqrt{x^2-9}\,dx$.

Solution.

$$\begin{aligned}
\int x\sqrt{x^2-9}\,dx &= \int \frac{x\sqrt{u}}{2x}\,du && u = x^2-9 \iff du = 2x\,dx \\
&= \frac{1}{2}\int \sqrt{u}\,du \\
&= \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} + C \\
&= \frac{1}{3}\left(x^2-9\right)^{3/2} + C
\end{aligned}$$

(iv) Compute $\displaystyle\int_0^3 \frac{x}{(1+x^2)^2}\,dx$.

Solution.

$$\begin{aligned}
\int_0^{\sqrt{3}} \frac{x}{(1+x^2)^2}\,dx &= \int_0^{\pi/3} \frac{\tan(\theta)\sec^2(\theta)}{\left[1+\tan^2(\theta)\right]^2}\,d\theta && x = \tan(\theta) \iff dx = \sec^2(\theta)\,d\theta \\
&= \int_0^{\pi/3} \frac{\tan(\theta)\sec^2(\theta)}{\sec^4(\theta)}\,d\theta \\
&= \int_0^{\pi/3} \frac{\tan(\theta)}{\sec^2(\theta)}\,d\theta \\
&= \int_0^{\pi/3} \frac{\sin(\theta)}{\cos(\theta)}\left(\cos^2(\theta)\right)\,d\theta \\
&= \int_0^{\pi/3} \sin(\theta)\cos(\theta)\,d\theta && u = \sin(\theta) \iff du = \cos(\theta)\,d\theta \\
&= \int_0^{\sqrt{3}/2} u\,du \\
&= \left[\frac{u^2}{2}\right]_0^{\sqrt{3}/2} \\
&= \frac{3}{4}
\end{aligned}$$

REMARK 2.1.5

You may need to complete the square before making a trigonometric substitution.

EXERCISE 2.1.6

Show that $3-2x-x^2 = 4-(x+1)^2$ by completing the square.

EXAMPLE 2.1.7

Compute $\displaystyle\int \frac{x}{(3-2x-x^2)^{3/2}}\,dx$.

Solution. Substitution: $x+1 = 2\sin(\theta) \iff dx = 2\cos(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{x}{(3-2x-x^2)^{3/2}}\,dx &= \int \frac{x}{\left[4-(x+1)^2\right]^{3/2}}\,dx \\
&= \int \frac{[2\sin(\theta)-1](2\cos(\theta))}{\left[4-4\sin^2(\theta)\right]^{3/2}}\,d\theta \\
&= \frac{1}{4}\int \frac{[2\sin(\theta)-1]\,(\cos(\theta))}{\left[\cos^2(\theta)\right]^{3/2}}\,d\theta \\
&= \frac{1}{4}\int \frac{2\sin(\theta)-1}{\cos^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int \frac{2\sin(\theta)}{\cos^2(\theta)} - \frac{1}{\cos^2(\theta)}\,d\theta \\
&= \frac{1}{4}\int 2\tan(\theta)\sec(\theta) - \sec^2(\theta)\,d\theta \\
&= \frac{1}{4}\left[2\sec(\theta)-\tan(\theta)\right] + C \\
&= \frac{1}{4}\left[\frac{4}{\sqrt{4-(x+1)^2}} - \frac{(x+1)}{\sqrt{4-(x+1)^2}}\right] + C
\end{aligned}$$

Where we substituted $x+1 = 2\sin(\theta) \iff \sin(\theta) = \dfrac{x+1}{2} \implies \sec(\theta) = \dfrac{2}{\sqrt{4-(x+1)^2}}$ and $\tan(\theta) = \dfrac{x+1}{\sqrt{4-(x+1)^2}}$ in the last step.

2.2 Integration by Parts

Let u and v be functions of x. From the product rule, we know

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides gives:

$$\int \frac{d}{dx}[uv]\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

Omit dx's to make

$$uv = \int u\,dv - \int v\,du$$

So, we get:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Strategy: When integrating the product of two functions, pick one to integrate (call it dv), and one to differentiate (call it u).

- Pick dv to be the most difficult function you know how to integrate.
- Pick u so that it gets simpler when differentiated.

Or, use ILATE: Pick $u =$ the first function in the list:

- I: Inverse trigonometric functions

- L: Logarithmic functions
- A: Algebraic functions (powers of x)
- T: Trigonometric functions
- E: Exponential functions

EXAMPLE 2.2.1

(i) Compute $\int x^2 \ln(x)\, dx$.

Solution. Let $u = \ln(x)$ and $dv = x^2\, dx$, so we have $du = \frac{1}{x}\, dx$ and $v = \frac{x^3}{3}$.

$$\begin{aligned}\int x^2 \ln(x)\, dx &= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\left(\frac{1}{x}\right) dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\, dx \\ &= \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C\end{aligned}$$

(ii) Compute $\int xe^x\, dx$.

Solution. Let $u = x$ and $dv = e^x\, dx$, so we have $du = dx$ and $v = e^x$.

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C$$

(iii) Compute $\int_0^\pi x\cos(x)\, dx$.

Solution. Let $u = x$ and $dv = \cos(x)dx$, so we have $du = dx$ and $v = \sin(x)$.

$$\int_0^\pi x\cos(x)\, dx = [x\sin(x)]_0^\pi - \int_0^\pi \sin(x)\, dx = [\cos(x)]_0^\pi = \cos(\pi) - \cos(0) = -1 - 1 = -2$$

(iv) Compute $\int \ln(x)\, dx$.

Solution. Sometimes, we don't want to integrate any part! Let $u = \ln(x)$ and $dv = dx$, so we have $du = \frac{1}{x}\, dx$ and $v = x$.

$$\int \ln(x)\, dx = x\ln(x) - \int \frac{x}{x}\, dx = x\ln(x) - x + C$$

This method also works for $\int \arctan(x)\, dx$ and $\int \arccos(x)\, dx$, etc.

(v) Compute $\int x^2\cos(x)\, dx$.

Solution. We may need to apply it more than once! Let $u = x^2$ and $dv = \cos(x)\, dx$, so we have $du = 2x\, dx$ and $v = \sin(x)$.

$$\int x^2\cos(x)\, dx = x^2\sin(x) - \int 2x\sin(x)\, dx$$

Let $u = 2x$ and $dv = \sin(x)\, dx$, so we have $du = 2\, dx$ and $v = -\cos(x)$.

$$\begin{aligned}&= x^2\sin(x) - \left[-2x\cos(x) - \int -2\cos(x)\, dx\right] \\ &= x^2\sin(x) + 2x\cos(x) - \int 2\cos(x)\, dx \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\end{aligned}$$

(vi) Compute $\int e^x \cos(x)\,dx$.

Solution. And sometimes, we don't integrate at all! Let $u = \cos(x)$ and $dv = e^x\,dx$, so we have $du = -\sin(x)\,dx$ and $v = e^x$.

$$I = \int e^x \cos(x)\,dx = e^x \cos(x) + \int e^x \sin(x)\,dx$$

Let $u = \sin(x)$ and $dv = e^x\,dx$, so we have $du = \cos(x)\,dx$ and $v = e^x$.

$$I = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\,dx = e^x \cos(x) + e^x \sin(x) - I$$

So, $2I = e^x \cos(x) + e^x \sin(x)$, therefore

$$I = \frac{e^x \cos(x) + e^x \sin(x)}{2} + C$$

Neat!

(vii) Compute $\int x^3 \cos(x^2)\,dx$.

Solution. Sometimes, a combination of methods is needed.

$$\begin{aligned}\int x^3 \cos(x^2)\,dx &= \int x^2 \cos(u)\frac{1}{2x}\,du \qquad\qquad u = x^2 \iff du = 2x\,dx \\ &= \frac{1}{2}\int x^2 \cos(u)\,du \\ &= \frac{1}{2}\int u\cos(u)\,du\end{aligned}$$

Now, do integration by parts with some unfortunate (but fine) letter choices! Let $u = u$ and $dv = \cos(u)\,du$, so we have $du = du$ and $v = \sin(u)$.

$$\begin{aligned}&= \frac{1}{2}u\sin(u) - \frac{1}{2}\int \sin(u)\,du \\ &= \frac{1}{2}u\sin(u) + \frac{1}{2}\cos(u) + C \\ &= \frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C\end{aligned}$$

2.3 Partial Fractions

Partial fractions are useful for evaluating $\int \frac{p(x)}{q(x)}\,dx$ where p and q are polynomials.

Overall idea: break a difficult integrand into many easy ones!

REMARK 2.3.1

We will assume the degree of the denominator is **larger** than the degree of the numerator. If not, use long division first!

Table 2.1: How to Break up Fractions: The Rules

If the denominator has:	Then we write:
(I) Distinct linear factors	One constant per factor
(II) A repeated linear factor	One constant per power
(III) Distinct irreducible quadratic factors	One **linear term** per factor
(IV) Repeated irreducible quadratic factors	One linear term per power

EXAMPLE 2.3.2: Decomposition Practice

(i) $\dfrac{1}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

(ii) $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$

(iii) $\dfrac{x^3+x+7}{x^2(x+1)^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{Ex+F}{x^2+1}$

(iv) $\dfrac{x^2+7}{(x-1)^3(x^2+3)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{Dx+E}{x^2+3} + \dfrac{Fx+G}{(x^2+3)^3}$

(v) $\dfrac{x^{10}+5}{(x+1)^3(x^2+1)} = \cdots$ use long division first, not partial fractions

REMARK 2.3.3: What integrals could we be left with after partial fractions?

Case 1: $\displaystyle\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$

Case 2: $\displaystyle\int \frac{A}{(ax+b)^n}\,dx = \left(\frac{A}{a}\right)\frac{(ax+b)^{-n+1}}{-n+1}$ where $n \neq 0, 1$.

Case 3: $\displaystyle\frac{Ax+B}{ax^2+bx+c} = \int \frac{Ax}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}\,dx$

Case 4: $\displaystyle\frac{Ax+B}{(ax^2+bx+c)^n}$

Note for Case 3 and Case 4, you may want to complete the square and use a trigonometric substitution. A regular substitution may also work.

EXAMPLE 2.3.4: Partial Fractions (Easy)

Using partial fractions, compute $\displaystyle\int \frac{x}{x^2-4x-5}\,dx$.

Solution. First, we break it up with partial fractions.

$$\frac{x}{x^2-4x-5} = \frac{x}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$$

Multiply both sides by the LHS denominator to get the following.

$$x = (x+1)(x-5)\left[\frac{A}{x+1} + \frac{B}{x-5}\right]$$
$$x = A(x-5) + B(x+1)$$

There are two ways we can solve for A and B.

(i) Linear Algebra!

$$x = Ax - 5A + Bx + B = (A+B)x + (-5A+B)$$

Therefore, $A+B=1$ and $B-5A=0$. Thus, $A = 1/6$ and $B = 5/6$.

(ii) Substitute in "nice" values for x.

$$\begin{aligned} x = 5: &\quad 5 = A(0) + B(6) \\ x = 1: &\quad -1 = A(-6) + B(0) \end{aligned}$$

Thus, $A = 1/6$ and $B = 5/6$.
Either way, we get

$$\frac{x}{(x+1)(x-5)} = \frac{1/6}{x+1} + \frac{5/6}{x-5}$$

So,

$$\int \frac{x}{x^2-4x-5}\,dx = \frac{1}{6}\int \frac{1}{x+1}\,dx + \frac{5}{6}\int \frac{1}{x-5}\,dx = \frac{1}{6}\ln|x+1| + \frac{5}{6}\ln|x-5| + C$$

EXAMPLE 2.3.5: Partial Fractions (Slightly Difficult)

Using partial fractions, compute $\displaystyle\int \frac{x+3}{x^4+9x^2}\,dx$.

Solution. First, we break it up with partial fractions.

$$\frac{x+3}{x^4+9x^2} = \frac{x+3}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$$

Multiply both sides by $x^2(x^2+9)$ to get the following.

$$\begin{aligned} x+3 &= x(x^2+9)A + (x^2+9)B + x^2(Cx+D) \\ x+3 &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2 \\ x+3 &= (A+C)x^3 + (B+D)x^2 + 9Ax + 9B \end{aligned}$$

Therefore, $A+C=0$, $B+D=0$, $9A=1$, and $9B=3$. Thus, $A = 1/9$, $B = 1/3$, $C = -1/9$, and $D = -1/3$. So,

$$\begin{aligned} \int \frac{x+3}{x^4+9x^2}\,dx &= \frac{1}{9}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{x^2}\,dx - \frac{1}{9}\int \frac{x}{x^2+9}\,dx - \frac{1}{3}\int \frac{1}{x^2+9}\,dx \\ &= \frac{1}{9}\ln|x| - \frac{1}{3x} - \frac{1}{9}\int \frac{x}{x^2+9}\,dx - \frac{1}{3}\left[\frac{1}{3}\arctan\left(\frac{x}{3}\right)\right] \end{aligned}$$

where we computed $\displaystyle\int \frac{1}{x^2+9}\,dx$ with 2.3.6.

For $\displaystyle\int \frac{x}{x^2+9}\,dx$, use a substitution: $u = x^2+9 \iff du = 2x\,dx$.

$$\int \frac{x}{x^2+9}\,dx = \int \frac{x}{u}\frac{1}{2x}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{\ln|u|}{2} + C = \frac{\ln|x^2+9|}{2} + C$$

So, the final answer is:

$$\frac{1}{9}\ln|x| - \frac{1}{3x} - \frac{1}{18}\ln|x^2+9| - \frac{1}{9}\arctan\left(\frac{x}{3}\right) + C$$

REMARK 2.3.6: Useful Identity

$$\int \frac{1}{x^2+k^2}\,dx = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + C$$

EXAMPLE 2.3.7: Partial Fractions (Long Division)

Compute $\displaystyle\int \frac{x^3-2x}{x^2+3x+2}\,dx$.
Solution. Using long division, we get

$$\int x - 3 + \frac{5x+6}{x^2+3x+2}\,dx$$

Now,

$$\frac{5x+6}{x^2+3x+2} = \frac{5x+6}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Therefore, $5x+6 = A(x+2) + B(x+1) = x(A+B) + (2A+B)$. By inspection, we get $B = 4$ and $A = 1$. Thus, the integral is:

$$\int x - 3 + \frac{1}{x+1} + \frac{4}{x+2}\,dx = \frac{x^2}{2} - 3x + \ln|x+1| + 4\ln|x+2| + C$$

2.4 Improper Integrals

So far, we have only examined integrals of continuous, or at least bounded functions. Let's see how to deal with a more general collection of functions!

In particular, we will examine two types:

(1) Continuous functions over infinite intervals

(2) Functions with infinite discontinuities

In particular:

- Type I: Infinite Intervals. Integrals of the form $\displaystyle\int_{-\infty}^{a} f(x)\,dx,\ \int_{a}^{\infty} f(x)\,dx,$ or $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx.$
- Type II: Infinite Discontinuity. For example, $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ as there is an issue at $x = 0$.

In all cases, the idea is to replace the problematic point with a letter and take a limit.

Let's see them in more detail now!

Type I

We replace the infinite endpoint with a letter and take a limit

- $\displaystyle\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty}\int_{b}^{a} f(x)\,dx$
- $\displaystyle\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$

- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b_1\to-\infty}\int_{b_1}^{0} f(x)\,dx + \lim_{b_2\to\infty}\int_{0}^{b_2} f(x)\,dx$

REMARK 2.4.1

Don't use $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$. This is called the "Cauchy Principal Value" and it is something else!

We say that the integral **converges** if all the limits exist (and are finite). The integral **diverges** if even one limit does not exist (or is $\pm\infty$).

EXAMPLE 2.4.2: Type I Integrals

Evaluate the following or show they diverge.

(i) $\displaystyle\int_{2}^{\infty} \frac{1}{x^2}\,dx$

Solution.

$$\int_{2}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_{2}^{b} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_2^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{2}\right) = \frac{1}{2}$$

Thus, the integral converges.

(ii) $\displaystyle\int_{-\infty}^{\infty} \sin(x)\,dx$

Solution.

$$\int_{-\infty}^{\infty} \sin(x)\,dx = \lim_{b_1\to-\infty}\int_{b_1}^{0} \sin(x)\,dx + \lim_{b_2\to\infty}\int_{0}^{b_2} \sin(x)\,dx$$

Let's evaluate the first one:

$$\lim_{b_1\to-\infty}\int_{b_1}^{0} \sin(x)\,dx = \lim_{b_1\to-\infty}\left[-\cos(x)\right]_{b_1}^{0} = \lim_{b_1\to-\infty}\left[-\cos(0)+\cos(b_1)\right]$$

which does not exist. Therefore, this integral diverges, there is no need to check the second limit!

(iii) $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

Solution.

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_{0}^{b} \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty}\left[\arctan(x)\right]_0^b \\ &= \lim_{b\to\infty}\left[\arctan(b)-\arctan(0)\right] \\ &= \frac{\pi}{2}-0 \\ &= \frac{\pi}{2}\end{aligned}$$

Thus, the integral converges.

Question: For which $p\in\mathbb{R}$ does $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converge?

Let's find out!

Proof of 2.4.3

<u>Case 1</u>: $p > 1$.

$$\lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right) = \frac{1}{p-1}$$

since $-p+1 < 0$, so $b^{-p+1} \to 0$. So, the integral converges if $p > 1$.
<u>Case 2</u>: $p < 1$. The calculation is the same as Case 1, until:

$$\lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = \infty$$

since $-p+1 > 0$, so $b^{-p+1} \to \infty$. So, the integral diverges if $p < 1$.
<u>Case 3</u>: $p = 1$.

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln|x|]_1^b = \lim_{b\to\infty}(\ln|b| - \ln|1|) = \infty$$

So, the integral diverges if $p = 1$.

Therefore, we have proven:

THEOREM 2.4.3: p-Integrals

The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ *converges if and only if* $p > 1$. *If* $p > 1$, $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$.

Next, let's examine some properties of Type I improper integrals.

THEOREM 2.4.4: Properties of Type I Improper Integrals

Suppose $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ *both converge.*

(1) $\int_a^\infty cf(x)\,dx$ *converges for any* $c \in \mathbb{R}$, *and*

$$\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$$

(2) $\int_a^\infty f(x)+g(x)\,dx$ *converges, and*

$$\int_a^\infty f(x)+g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

(3) If $f(x) \leqslant g(x)$ *for all* $x \geqslant a$, *then*

$$\int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$$

(4) If $a < c < \infty$, *then* $\int_c^\infty f(x)\,dx$ *converges, and*

$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Evaluating integrals in general is hard, and determining if an improper integral converges may be even harder! However, we do have a way of comparing a difficult integral to a simpler one (for example, a p-Integral!).

The Comparison Theorem (For Type I)

THEOREM 2.4.5: Comparison Test for Type I Improper Integrals

Assume $0 \leqslant g(x) \leqslant f(x)$ *for all* $x \geqslant a$ *and that both* f *and* g *are continuous on* $[a, \infty)$.

(1) If $\displaystyle\int_a^\infty f(x)\,dx$ *converges, then so does* $\displaystyle\int_a^\infty g(x)\,dx$.

(2) If $\displaystyle\int_a^\infty f(x)\,dx$ *diverges, then so does* $\displaystyle\int_a^\infty g(x)\,dx$.

EXAMPLE 2.4.6

Determine if $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges or diverges.

Solution. Note for $x > 1$, $0 \leqslant e^{-x^2} < e^{-x}$, by comparison since $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges, so does $\displaystyle\int_0^\infty e^{-x^2}\,dx$.

REMARK 2.4.7

It doesn't matter that the inequality doesn't hold for $0 \leqslant x < 1$, since $\int_0^1 e^{-x^2}\,dx$ is not improper and so converges.

EXAMPLE 2.4.8

Determine of the following integrals converge or diverge.

(i) $\displaystyle\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$.

Solution. Note that $0 \leqslant \dfrac{x}{(x^2+2)^2} \leqslant \dfrac{x}{x^4} = \dfrac{1}{x^3}$, for $x \geqslant 1$.

Since $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges (p-integral), so does $\displaystyle\int_1^\infty \frac{x}{(x^2+2)^2}\,dx$, by comparison.

(ii) $\displaystyle\int_1^\infty \frac{2x^2}{x^3-x+1}\,dx$.

Solution. Note that $\dfrac{2x^2}{x^3-x+1} \geqslant \dfrac{2x^2}{x^3+1} \geqslant \dfrac{2x^2}{x^3+x^3} = \dfrac{2}{2x} = \dfrac{1}{x} \geqslant 0$, for $x \geqslant 1$.

Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges (p-integral), so does $\displaystyle\int_1^\infty \frac{2x}{x^3-x+1}\,dx$, by comparison.

(iii) $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$.

Solution. Note that $\dfrac{1+e^{-x}}{x} \geqslant \dfrac{1}{x} \geqslant 0$, for $x \geqslant 1$.

Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, so does $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$, by comparison.

(iv) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$.

Solution. Note that $0 \leqslant \dfrac{e^x}{e^{2x}+3} \leqslant \dfrac{e^x}{e^{2x}} = e^{-x}$, for $x \geqslant 0$.

Since $\displaystyle\int_0^\infty e^{-x}\,dx$ converges, so does $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$, by comparison.

The comparison theorem is fantastic, but it only works on non-negative functions. How can we deal with negative functions?

We can use absolute values!

DEFINITION 2.4.9: Absolute convergence

Let f be integrable on $[a, b)$ for all $b \geqslant a$. We say that the improper integral $\displaystyle\int_a^\infty f(x)\,dx$ **converges absolutely** if $\displaystyle\int_a^\infty |f(x)|\,dx$ converges.

THEOREM 2.4.10: Absolute Convergence Theorem (ACT)

Let f be integrable on $[a, b]$ for all $b \geqslant a$. Then $|f|$ is also integrable on $[a, b]$ for all $b \geqslant a$. Moreover, if we assume that $\displaystyle\int_a^\infty |f(x)|\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
In particular, if $|f(x)| \leqslant g(x)$ for all $x \geqslant a$, both f and g are integrable on $[a, b]$ for all $b \geqslant a$, and if $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.

Proof of 2.4.10

Suppose $\int_a^\infty |f(x)|\,dx$ converges. Then so does

$$\int_a^\infty 2|f(x)|\,dx.$$

Note that

$$0 \leqslant f(x) + |f(x)| \leqslant 2|f(x)|,$$

so by comparison,

$$\int_a^\infty f(x) + |f(x)|\,dx$$

converges. But

$$\int_a^\infty f(x)\,dx = \int_a^\infty f(x) + |f(x)|\,dx - \int_a^\infty |f(x)|\,dx$$

converges, since both components do.

EXAMPLE 2.4.11

Determine if $\displaystyle\int_1^\infty \frac{\sin(x)}{x^2+1}\,dx$ converges or diverges.

Solution. We can't use comparison directly since $\sin(x) < 0$ sometimes. But,

$$0 \leqslant \left|\frac{\sin(x)}{x^2+1}\right| \leqslant \frac{1}{x^2+1} \leqslant \frac{1}{x^2}$$

for $x \geqslant 1$, and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges. So by ACT, $\displaystyle\int_1^\infty \frac{\sin(x)}{1+x^2}\,dx$ converges.

Now, let's switch to Type II improper integrals. We will see how to deal with them, but we won't go as deeply into them.

Type II

Consider $\displaystyle\int_a^b f(x)\,dx$.

- If f has an infinite discontinuity at $x = a$, then we use
$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$
- If f has an infinite discontinuity at $x = b$, then we use
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_t^b f(x)\,dx$$
- If f is not continuous at c, $a < c < b$, then we write
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
and use the limits as the previous cases.

Again, if all limit(s) exist, then we say the integral **converges**. If even one limit does not exist, then the integral **diverges**.

EXAMPLE 2.4.12

Determine of the following integrals converge or diverge.

(i) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx.$

Solution. There is a problem at $x = 1$. So,
$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} \left[-2\sqrt{1-x}\right]_0^t = \lim_{t\to 1^-} [-2\sqrt{1-t} + 2\sqrt{1}] = 2$$
Thus, the integral converges.

(ii) $\displaystyle\int_2^3 \frac{x}{(x^2-4)^2}\,dx$

Solution. There is a problem at $x = 2$, but let's make a substitution first! Let $u = x^2 - 4$, so $du = 2x\,dx$. If $x = 2$, $u = 0$, and if $x = 3$, $u = 5$. So,
$$\int_2^3 \frac{x}{(x^2-4)^2}\,dx = \int_0^5 \frac{x}{u^2}\frac{1}{2x}\,du = \lim_{t\to 0^+} \frac{1}{2}\int_t^5 \frac{1}{u^2}\,du = \lim_{t\to 0^+} \left[-\frac{1}{2u}\right]_t^5 = \lim_{t\to 0^+} \left[-\frac{1}{10} + \frac{1}{2t}\right] = \infty$$
Thus, the integral diverges.

(iii) $\displaystyle\int_0^3 \frac{1}{(x-2)^{1/3}}\,dx.$

Solution. There is a problem at $x = 2$.

$$\begin{aligned}
\int_0^3 \frac{1}{(x-2)^{1/3}}\,dx &= \int_0^2 \frac{1}{(x-2)^{1/3}}\,dx + \int_2^3 \frac{1}{(x-2)^{1/3}}\,dx \\
&= \lim_{t_1\to 2^-}\int_0^{t_1} \frac{1}{(x-2)^{1/3}}\,dx + \lim_{t_2\to 2^+}\int_{t_2}^3 \frac{1}{(x-2)^{1/3}}\,dx \\
&= \lim_{t_1\to 2^-}\left[\frac{3}{2}(x-2)^{2/3}\right]_0^{t_1} + \lim_{t_2\to 2^+}\left[\frac{3}{2}(x-2)^{2/3}\right]_{t_2}^3 \\
&= \lim_{t_1\to 2^-}\frac{3}{2}\left[(t_1-2)^{2/3}-(0-2)^{2/3}\right] + \lim_{t_2\to 2^+}\frac{3}{2}\left[(3-2)^{2/3}-(t_2-2)^{2/3}\right] \\
&= -\frac{3}{2}(-2)^{2/3} + \frac{3}{2}
\end{aligned}$$

Thus, the integral converges.

EXERCISE 2.4.13

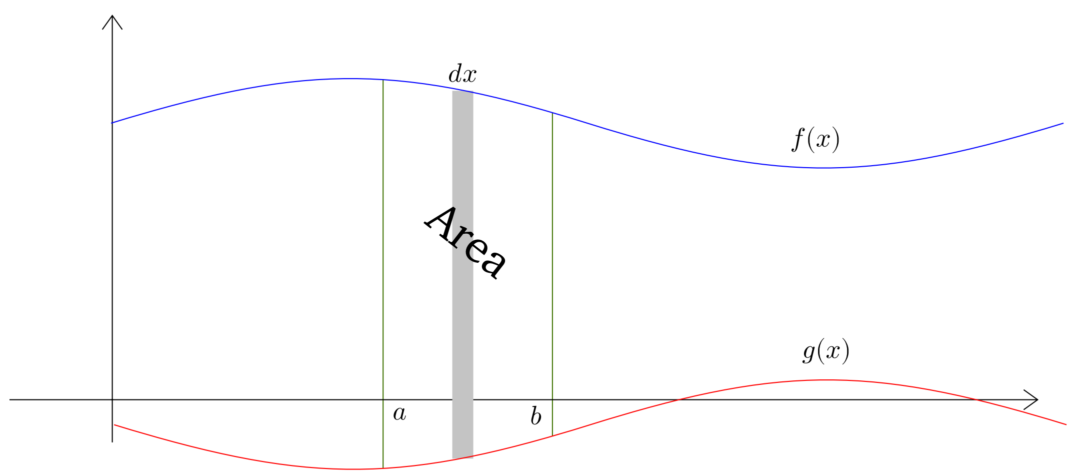
Show $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges if and only if $p < 1$.

Chapter 3

Applications of Integration

3.1 Area Between Curves

Suppose we want to calculate the area of the region *between* f and g.



This is the area under f *minus* the area "under" (between g and the x-axis) g.

Using the same ideas as before, we divide the region into infinitely many infinitely thin rectangles (each with width $dx \approx \Delta x$) and integrate!

So, for $f \geqslant g$, the area bounded by f and g from $x = a$ to $x = b$ is

$$\int_a^b f(x) - g(x)\, dx$$

EXAMPLE 3.1.1

Find the area between $f(x) = x^2$ and $g(x) = x$ from $x = 1$ to $x = 3$.
Solution. Since $x^2 \geqslant x$ for $x \in [1, 3]$, we get

$$\text{Area} = \int_1^3 x^2 - x\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 = \left(\frac{27}{3} - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{14}{3}$$

REMARK 3.1.2

You should always get a positive answer! If your answer is negative you should go check your work! Note that the "upper" curve may change over the interval.

Actual Formula: Area between f and g from $x = a$ to $x = b$ is

$$\int_a^b |f(x) - g(x)|\, dx$$

So, we should split up the interval $[a, b]$ to eliminate the absolute value.

Remember, $\displaystyle\int_a^b$ "upper" $-$ "lower" dx.

EXAMPLE 3.1.3

Find the area enclosed by $f(x) = 1 - x^2$ and $g(x) = x^2$.

Solution. First, we need to find the intersection points: $1 - x^2 = x^2 \iff 1 = 2x^2 \iff x = \pm\dfrac{1}{\sqrt{2}}$.

If $x \in \left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$, then $1 - x^2 \geqslant x^2$, so we get:

$$\begin{aligned}
\text{Area} &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - x^2) - x^2\, dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 1 - 2x^2\, dx \\
&= \left[x - \frac{2}{3}x^3\right]_{-1/\sqrt{2}}^{1/\sqrt{2}} \\
&= \left(\frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3\right) - \left(-\frac{1}{\sqrt{2}} - \frac{2}{3}\left(-\frac{1}{\sqrt{2}}\right)^3\right) \\
&= \frac{2}{\sqrt{2}} - \frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^3 \\
&= \frac{2}{\sqrt{2}} - \frac{2}{3\sqrt{2}} \\
&= \frac{4}{3\sqrt{2}}
\end{aligned}$$

EXAMPLE 3.1.4

Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ from $x = 0$ to $x = \pi$.

Solution. From $x = 0$ to $x = \pi/4$: $\cos(x) \geqslant \sin(x)$. From $\pi/4$ to $x = \pi$: $\sin(x) \geqslant \cos(x)$. So, the area is:

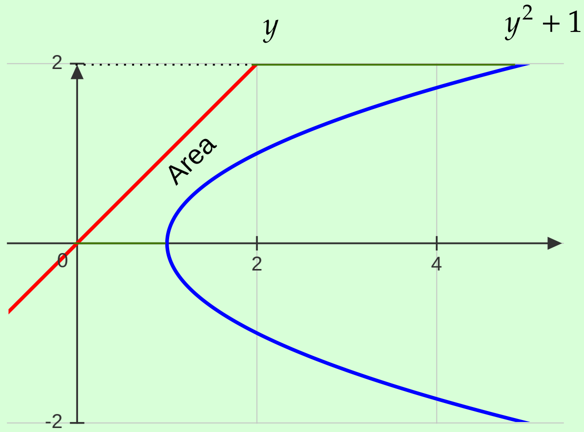
$$\begin{aligned}
A &= \int_0^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi} \sin(x) - \cos(x)\,dx \\
&= \left[\sin(x) + \cos(x)\right]_0^{\pi/4} + \left[-\cos(x) - \sin(x)\right]_{\pi/4}^{\pi} \\
&= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin(0) - \cos(0) - \cos(\pi) - \sin(\pi) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 + 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\
&= 2\sqrt{2}
\end{aligned}$$

We can use the same ideas to compute areas when x is a function of y, except we use

"outer" − "inner" or "right" − "left"

EXAMPLE 3.1.5

Find the area between $x = y^2 + 1$ and $y = x$ from $y = 0$ to $y = 2$.



Solution. For $y \in [0, 2]$: $y^2 + 1 \geqslant y$, so:

$$\begin{aligned}
A &= \int_0^2 y^2 + 1 - y\,dy \\
&= \left[\frac{y^3}{3} + y - \frac{y^2}{2}\right]_0^2 \\
&= \frac{8}{3} + 2 - 2 \\
&= \frac{8}{3}
\end{aligned}$$

3.2 Volumes of Revolution

As we did for areas between curves, we can use our knowledge of integrals to compute the volume of certain objects: ones obtained by rotating a region about a horizontal or vertical line!

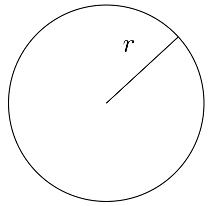


Figure 3.1: Area of a disk with radius r: πr^2

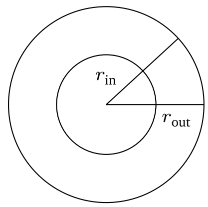


Figure 3.2: Area of a washer with outer radius r_{out} and inner radius r_{in}: $\pi(r_{\text{out}})^2 - \pi(r_{\text{in}})^2$

REMARK 3.2.1

For more general shapes, we need multivariable methods. These are explored in MATH 237!

Let's get a formula!

Areas:

- Area of one infinitesimally thin rectangle: $f(x)\,dx$
- Overall area: $\displaystyle\int_a^b f(x)\,dx$

Volumes (rotate f around x-axis):

- Volume of one infinitesimally thin slice: $A(x)\,dx$
- Overall volume: $\displaystyle\int_a^b A(x)\,dx$

So, we just need to determine $A(x)$ in each case! There are a few different methods we will use.

The two main methods are:

(I) Washers/disks (i.e., cross-sections)

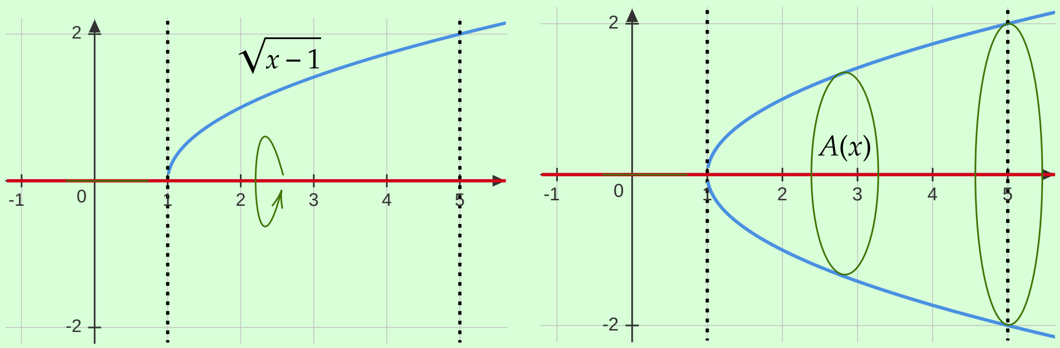
(II) Cylindrical shells

Method 1: Washers/disks

First, let's recall the area formulas:

EXAMPLE 3.2.2

Find the volume of the solid obtained by rotating $f(x) = \sqrt{x-1}$ about the x-axis from $x = 1$ to $x = 5$.



Solution. The cross-section is a disk with radius $\sqrt{x-1}$. So

$$A(x) = \pi\left(\sqrt{x-1}\right)^2 = \pi(x-1)$$

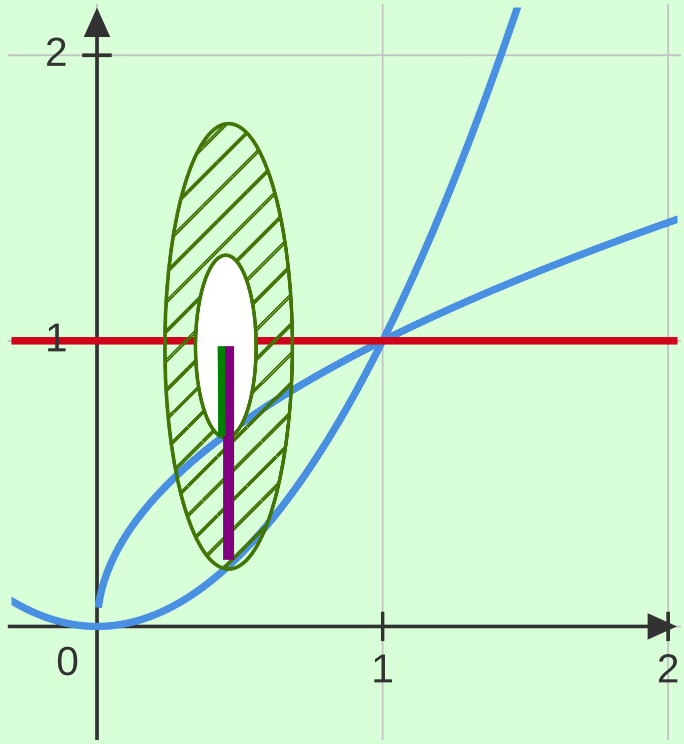
and

$$\begin{aligned} \text{Volume} &= \int_1^5 A(x)\,dx \\ &= \int_1^5 \pi(x-1)\,dx \\ &= \pi\left[\frac{x^2}{2} - x\right]_1^5 \\ &= \pi\left(\frac{25}{2} - 5 - \frac{1}{2} + 1\right) \\ &= 8\pi \end{aligned}$$

Note that you don't need to draw the full 3-D image, just one area slice is enough!

EXAMPLE 3.2.3

Rotate the area between $y = \sqrt{x}$ and $y = x^2$ about the line $y = 1$.



Solution. The cross-section is a washer with $r_{\text{out}} = 1 - x^2$ and $r_{\text{in}} = 1 - \sqrt{x}$. So,

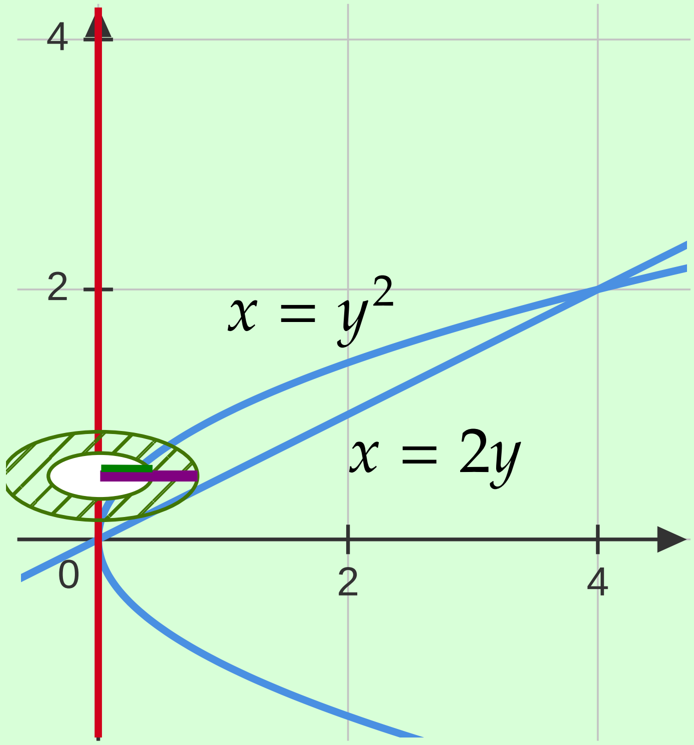
$$\begin{aligned} A(x) &= \pi(r_{\text{out}})^2 - \pi(r_{\text{in}})^2 \\ &= \pi\left[(1-x^2)^2 - (1-\sqrt{x})^2\right] \\ &= \pi(1 - 2x^2 + x^4 - 1 + 2\sqrt{x} - x) \\ &= \pi(x^4 - 2x^2 + 2\sqrt{x} - x) \end{aligned}$$

and

$$\begin{aligned} \text{Volume} &= \int_0^1 A(x)\,dx \\ &= \pi\int_0^1 x^4 - 2x^2 + 2x^{1/2} - x\,dx \\ &= \pi\left[\frac{x^5}{5} - \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 \\ &= \pi\left(\frac{1}{5} - \frac{2}{3} + \frac{4}{3} - \frac{1}{2}\right) \\ &= \frac{11\pi}{30} \end{aligned}$$

EXAMPLE 3.2.4

Rotate the region between $x = y^2$ and $x = 2y$ about the y-axis.



Solution. First, points of intersection: $y^2 = 2y \implies y = 0, 2$. The cross-section is a washer with $r_{\text{out}} = 2y$ and $r_{\text{in}} = y^2$. So,

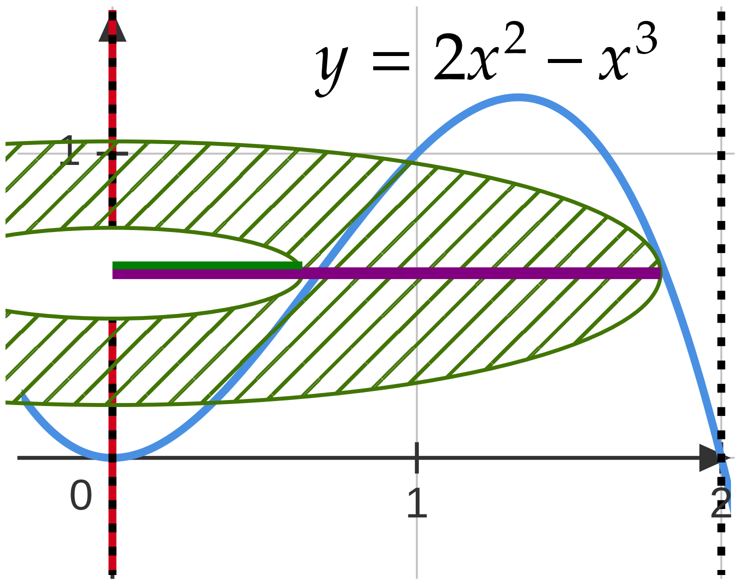
$$\begin{aligned} A(y) &= \pi\left(2y\right)^2 - \pi\left(y^2\right)^2 \\ &= \pi\left(4y^2 - y^4\right) \end{aligned}$$

and

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 4y^2 - y^4 \, dy \\ &= \text{exercise} \\ &= \frac{64\pi}{15} \end{aligned}$$

We can see that washers/disks arise when we rotate functions of x about a horizontal line or functions of y about a vertical line.

We can't use it all the time though, for example: rotate the region below $y = 2x^2 - x^3$ about the y-axis from $x = 0$ to $x = 2$.

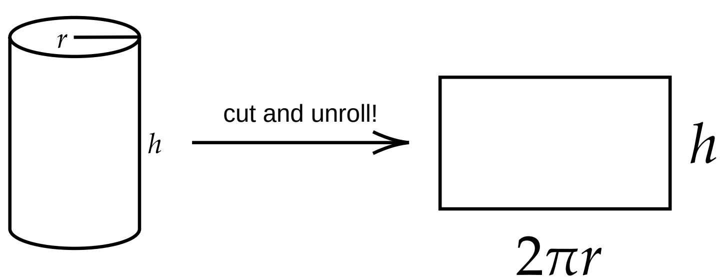


We need another method!

Method 2: Cylindrical shells

Instead of using cross-sections, we can use cylindrical shells (think soup can labels) to divide the volume up!

Q: What is the area of a cylindrical shell with base radius r and height h?

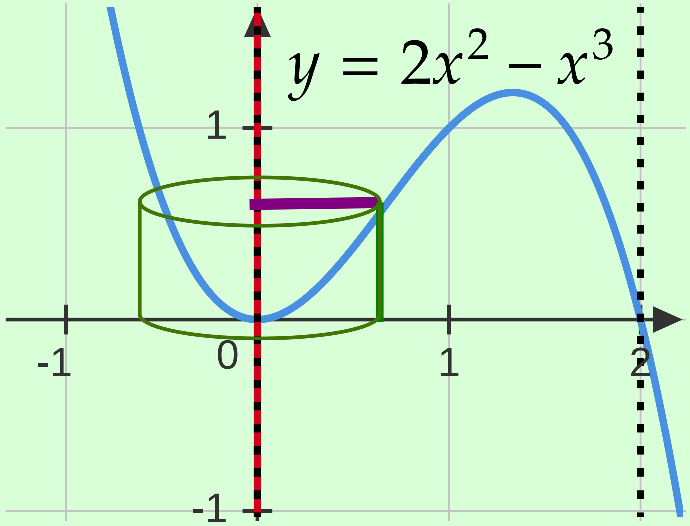


So area is $2\pi rh$. So, we only need to determine r and h!

Back to the difficult example:

EXAMPLE 3.2.5

Rotate the region below $y = 2x^2 - x^3$ about the y-axis from $x = 0$ to $x = 2$.



Solution. The area is a cylindrical shell with radius $r = x$ and height $h = 2x^2 - x^3$. So,

$$A(x) = 2\pi x \left(2x^2 - x^3\right)$$

and

$$\text{Volume} = \int_0^2 2\pi x \left(2x^2 - x^3\right)\, dx$$

EXAMPLE 3.2.6

Find the volume in each case.

(i) Rotate the region between $y = x^2$ and $y = 6x - x^2$ about the y-axis.
Solution. Intersection points: $x^2 = 6x - 2x^2 \implies x = 0, 2$

- $r = x$
- $h = 6x - 2x^2 - x^2 = 6x - 3x^2$

$$A(x) = 2\pi x \left(6x - 3x^2\right)$$

$$\begin{aligned} V &= \int_0^2 2\pi x \left(6x - 3x^2\right)\, dx \\ &= 2\pi \int_0^2 6x^2 - 3x^3\, dx \\ &= 2\pi \left[2x^3 - \frac{3}{4}x^4\right]_0^2 \\ &= 2\pi\,(16 - 12) \\ &= 8\pi \end{aligned}$$

(ii) Rotate the region bounded by $y = 4x - x^3$ and $y = 3$ about $x = 1$.
Solution. Intersection points: $4x - x^2 = 3 \implies x = 1, 3$

- $r = x - 1$
- $h = 4x - x^3 - 3$

$$A(x) = 2\pi(x - 1)(4x - x^2 - 3)$$

$$\begin{aligned} V &= \int_1^3 2\pi(x-1)(4x-x^2-3)\,dx \\ &= \text{exercise} \\ &= \frac{8\pi}{3} \end{aligned}$$

We will now look at some more practice examples, but first a handy table:

	Functions of x	Functions of y
Vertical line	Cylindrical shells	Washers/disks
Horizontal line	Washers/disks	Cylindrical shells

EXAMPLE 3.2.7: More Practice

Set up, but don't evaluate the integral(s) that would give the desired volume.

(i) Rotate the region bounded by $xy = 1$, $x = 0$, $y = 1$, $y = 3$, around the x-axis.

Solution 1. Cylindrical shell:

- $r = y$
- $h = 1/y$

$$A(y) = 2\pi y \left(\frac{1}{y}\right) = 2\pi$$

$$V = \int_1^3 2\pi\,dy = 4\pi$$

Solution 2. For fun, can we do it using washers/disks?

A: Yes! Just work with x and not y.

- $r_{\text{out}} = \begin{cases} 3 & 0 \leqslant x \leqslant 1/3 \\ 1/x & 1/3 \leqslant x \leqslant 1 \end{cases}$
- $r_{\text{in}} = 1$

$$V = \int_0^{1/3} \pi(3)^2 - \pi(1)^2\,dx + \int_{1/3}^1 \pi\left(\frac{1}{x}\right)^2 - \pi(1)^2\,dx$$

This method was more difficult, because r_{out} changes, but it is still valid!

(ii) Region bounded by $x = (y-1)^2$, $x = y+1$, about $x = -1$.

Solution. Points of intersection:

$$(y-1)^2 = y+1 \implies y = 0, 3$$

- $r_{\text{out}} = y + 1 - (-1) = y + 2$
- $r_{\text{in}} = (y-1)^2 - (-1) = (y-1)^2 + 1$

$$V = \pi\int_0^3 (y+2)^2 - \left[(y-1)^2+1\right]^2\,dy$$

Chapter 4

Differential Equations

4.1 Introduction to Differential Equations

DEFINITION 4.1.1: Ordinary Differential Equation

An equation containing derivatives of a dependent variable (i.e., function) $y = f(x)$ is called an **ordinary differential equation** (ODE).

REMARK 4.1.2

To contrast, there are also partial differential equations for multivariable functions.

EXAMPLE 4.1.3: Ordinary Differential Equations

- $y' + 2y = e^x$
- $y'' + y' + y = 0$
- $x^2y' + y = 31$

DEFINITION 4.1.4: Order

The **order** of an ODE is the order of the highest derivative that appears.

EXAMPLE 4.1.5: Order

- $y'' + y^3 = 0$: order 2
- $x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} = y$: order 2
- $y' + y = \sin(x)$: order 1

DEFINITION 4.1.6: Linear

An ODE is called **linear** if it contains only linear functions in y, y', y'', etc.

EXAMPLE 4.1.7: Linear and Non-linear ODEs

- $3y'' + 2x^3y = \cos(x)$: linear
- $y^2 + y' = 0$: not linear (y^2)
- $yy' = 0$: not linear

DEFINITION 4.1.8: General Solution

The **general solution** of an ODE is the collection of all possible solutions including arbitrary constants.

DEFINITION 4.1.9: Particular Solution

A **particular solution** is a solution in which all arbitrary constants have been determined.

DEFINITION 4.1.10: Initial Conditions

To get a particular solution, we would need some additional info, like values of y, y', y'', etc. for certain x-values. These are called **initial conditions**.

An ODE together with initial conditions is called an **initial value problem** (IVP).

In general, solving ODEs is difficult.

EXAMPLE 4.1.11

- $y' = x - y^2$: impossible to solve
- $y' = y - x^2$: easy to solve (next week!)

Soon, we will learn some techniques to solve certain ODEs, but for now we can only find some simple solutions.

EXAMPLE 4.1.12

What constant functions satisfy
$$y' = y^3 + 2y^2 - 80y$$

Solution. If $y = C$, a constant, then $y' = 0$, so we can get

$$0 = C^3 + 2C^2 - 80C = C(C + 10)(C - 8)$$

Therefore, $C = 0, -10, 8$. C is also known as **equilibrium solutions**.

4.2 Separable Differential Equations

In this course, we will consider first-order ODEs that can be written in the form

$$y' = f(x, y)$$

That is, we can solve for y'.

DEFINITION 4.2.1: Separable

A **separable** ODE is a first-order ODE that can be written as $\frac{dy}{dx} = g(y)h(x)$; that is, we can factor the RHS into a product of functions, one containing only x's and one containing only y's.

To solve a separable ODE, move $g(y)$ to the LHS and integrate both sides with respect to x.

$$\begin{aligned} \frac{dy}{dx} = g(y)h(x) &\implies \frac{1}{g(y)}\frac{dy}{dx} = h(x) \\ &\implies \int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int h(x)\,dx \\ &\implies \int \frac{1}{g(y)}\,dy = \int h(x)\,dx \end{aligned}$$

Now, integrate each side!

REMARK 4.2.2

What is going on in the last step? A substitution! Say $y = f(x)$, then $dy = f'(x)\,dx$. Therefore, the LHS is:

$$\int \frac{1}{g(f(x))} f'(x)\,dx = \int \frac{1}{g(y)}\,dy$$

EXAMPLE 4.2.3

Solve $\frac{dy}{dx} = \frac{x}{y}$, find the general solution.

Solution.

$$\frac{dy}{dx} = \frac{x}{y} \implies \int y\,dy = \int x\,dx \implies \frac{y^2}{2} = \frac{x^2}{2} + C$$

Solve for y if possible: $y = \pm\sqrt{x^2 + 2C}$ or $y = \pm\sqrt{x^2 + C_1}$.

EXAMPLE 4.2.4

Find the particular solution to the IVP $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$ with $y(0) = -1$.

Solution.

$$\int 2y - 2\,dy = \int 3x^2 + 4x + 2\,dx \implies y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Next, get C by using $y(0) = -1$.

$$(-1)^2 - 2(-1) = 0 + 0 + 0 + C \implies C = 3$$

So, $y^2 - 2y = x^3 + 2x^2 + 2x + 3$. We can solve for y if we complete the square on the LHS:

$$\begin{aligned} y^2 - 2y &= x^3 + 2x^2 + 2x + 3 \\ (y-1)^2 - 1 &= x^3 + 2x^2 + 2x + 3 \\ y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \end{aligned}$$

but, only one satisfies $y(0) = -1$:

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

REMARK 4.2.5

Watch out for dividing by zero! If you see a possible divide by zero (with y), then deal with that case separately.

EXAMPLE 4.2.6

Find the general solution to $\dfrac{dy}{dx} = \dfrac{y\cos(x)}{1+2y^2}$.

Solution. We get $\displaystyle\int \frac{1+2y^2}{y}\,dy = \int \cos(x)\,dx$ if $y \not\equiv 0$ "not identically" meaning not the constant function 0.

$$\int \frac{1}{y} + 2y\,dy = \sin(x) + C \implies \ln(y) + y^2 = \sin(x) + C$$

(can't solve for y).

But what if $y \equiv 0$? Then, $\dfrac{dy}{dx} = 0$, and the ODE becomes $0 = 0$ which is true for all x! So, $y \equiv 0$ is also a solution.

Therefore, the general solution is

$$\ln(y) + y^2 = \sin(x) + C \text{ or } y \equiv 0$$

Note that the $y \equiv 0$ solution (which is an equilibrium solution) is also called a **singular solution** since you can't get it by choosing C.

EXAMPLE 4.2.7

Find a particular solution for the IVP $\dfrac{dy}{dx} = xy$ with $y(0) = 1$.

Solution. We get $\displaystyle\int \frac{1}{y}\,dy = \int x\,dx$ if $y \not\equiv 0$, but note that $y(0) = 1$, so $y \not\equiv 0$!

$$\begin{aligned}
\ln|y| &= \frac{x^2}{2} + C \\
|y| &= e^{x^2/2} + C = e^C e^{x^2/2} \\
y &= \pm e^C e^{x^2/2} && \text{say } A = \pm e^C \\
y &= Ae^{x^2/2}
\end{aligned}$$

Use $y(0) = 1$ to get A: $1 = Ae^0 = A$, so $y = e^{x^2/2}$.

Sometimes an ODE isn't separable, but a substitution will make it separable.

Common substitutions: $V = y + x$, $V = {}^{y}/_{x}$, $V = y'$, etc.

EXAMPLE 4.2.8

Solve $\frac{dy}{dx} = (x+y)^2 - 1$

Solution. This ODE is not separable, but let $V = x + y$, so $V' = 1 + y'$ or $y' = V' - 1$. So,

$$V' - 1 = V^2 - 1 \implies V' = V^2$$

Now it's separable! Therefore,

$$\int \frac{1}{V^2}\,dV = \int dx$$

if $V \not\equiv 0$.

$$-\frac{1}{V} = x + C \implies V = -\frac{1}{x+C}$$

but $V = x + y$, so

$$x + y = -\frac{1}{x+C}$$

so we get

$$y = -x - \frac{1}{x+C}$$

What if $V \equiv 0$? Then $y = -x$, and $\frac{dy}{dx} = -1$, the ODE becomes $-1 = -1$ which is true for all x! So, $y = -x$ is another solution. Thus,

$$y = -x \text{ or } y = -x - \frac{1}{x+C}$$

Application: Mixing Problems

Suppose a tank has 1000 L of salt water at an initial concentration of 0.1 kg/L.

Salt water of concentration 0.3 kg/L flows in at a rate of 10 L/min. The solution is kept well-mixed, and drains out at the same rate.

Let $X(t) =$ the amount of salt in the tank at time t.

Then,

$$\begin{aligned}\frac{dx}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= (10\text{ L/min})(0.3\text{kg/L}) - (\text{conc. in tank})(10\text{ L/min})\end{aligned}$$

Therefore,

$$\begin{aligned}(\text{conc. in tank}) &= \frac{\text{Amount of salt}}{\text{volume}} \\ &= \frac{x}{1000} && \text{since volume doesn't change}\end{aligned}$$

So,

$$\begin{aligned}\frac{dx}{dt} &= 3 - \frac{x}{100} \\ &= \frac{300-x}{100} && \text{separable!}\end{aligned}$$

$$\int \frac{1}{300-x}\,dx = \int \frac{1}{100}\,dt \implies -\ln|300-x| = \frac{t}{100} + C$$

What is $X(0)$? Starts at 1000 L at 0.1 kg/L, so $X(0) = 100$ kg.

Find C:

$$-\ln|300-100| = C \implies C = -\ln(200)$$

So,

$$-\ln|300-x| = \frac{t}{100} - \ln(200)$$

Solve for x:

$$|300-x| = 200e^{-t/100}$$

but $300 - x \geqslant 0$ since $X(0) = 100$ and x is increasing to 300. So, we get:

$$X(t) = 300 - 200e^{-t/100}$$

4.3 Linear First-Order Differential Equations

The general form for a linear first-order ODE is

$$A(x)y' + B(x)y = C(x)$$

where $A(x) \not\equiv 0$. Or, dividing by $A(x)$, we can write it as:

$$y' + P(x)y = Q(x)$$

EXAMPLE 4.3.1: Preliminary Example

Solve $\frac{dy}{dx} + \frac{1}{x}y = 1$.
Solution. The trick is: multiply by x!

$$x\frac{dy}{dx} + y = x$$

Notice that now the LHS is the derivative of xy. So,

$$\frac{d}{dx}(xy) = x$$

Now, integrate both sides!

$$xy = \int x\,dx = \frac{x^2}{2} + C$$

Thus,

$$y = \frac{x}{2} + \frac{C}{x}$$

This is the general strategy: find a clever function to multiply the ODE by so that the LHS collapses into the derivative of a product. Then, we just need to integrate both sides and solve for y.

Let's find the useful function:

Say the ODE is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and the desired function is $\mu(x)$. Multiplying, we get

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

We **want** the LHS to be

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu'(x)y + \mu(x)\frac{dy}{dx}$$

Solving, we get:

$$\begin{aligned}
&\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)\frac{dy}{dx} + \mu'(x)y \\
&\implies \mu(x)P(x)y = \mu'(x)y && \text{should hold for all } y \\
&\implies \mu(x)P(x) = \mu'(x)
\end{aligned}$$

or $\mu(x)P(x) = \frac{d\mu}{dx}$ which is separable!

$$\int \frac{1}{\mu}\,d\mu = \int P(x)\,dx \implies \ln|\mu| = \int P(x)\,dx$$

Thus, the final form is

$$\boxed{\mu = e^{\int P(x)\,dx}}$$

REMARK 4.3.2

We can ignore the "$+C$" and absolute values since we only need to find one μ that works, not all of them.

So, we get an algorithm to solve a linear first-order ODE:

(I) Write it in the form

$$\frac{dy}{dx} = P(x)y = Q(x)$$

(II) Find

$$\mu(x) = e^{\int P(x)\,dx}$$

(III) Multiply the ODE by μ, collapse LHS into a product rule.

(IV) Integrate both sides (add $+C$) and solve for y.

EXAMPLE 4.3.3

Solve $\dfrac{dy}{dx} + 2xy = x$.
Solution.

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Multiply by e^{x^2}:

$$\begin{aligned} e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y &= xe^{x^2} \\ \implies \frac{d}{dx}\left[e^{x^2}y\right] &= xe^{x^2} \end{aligned}$$

Integrate:

$$\begin{aligned} e^{x^2}y &= \int xe^{x^2}\,dx & u = x^2 \iff du = 2x\,dx \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C \end{aligned}$$

So,

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C \implies y = \frac{1}{2} + \frac{C}{e^{x^2}}$$

EXAMPLE 4.3.4

Find a particular solution to $x^2\dfrac{dy}{dx} + 2xy = 1$ with $y(1) = 0$.
Solution. It's not in the correct form! First, divide by x^2:

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}$$

Now,

$$\begin{aligned}\mu(x) &= e^{\int 2/x\,dx}\\ &= e^{2\ln|x|}\\ &= e^{\ln(x)^2}\\ &= x^2\end{aligned}$$

So multiply by x^2, therefore our original ODE was actually what we wanted! Cool!

$$x^2\frac{dy}{dx} + 2xy = 1 \implies \frac{d}{dx}\left[x^2 y\right] = 1$$

Integrating gives

$$\begin{aligned}x^2 y &= x + C\\ y &= \frac{1}{x} + \frac{C}{x^2}\end{aligned}$$

Finally, we know $y(1) = 0$, so $0 = 1 + C \implies C = -1$. Thus,

$$y = \frac{1}{x} - \frac{1}{x^2}$$

REMARK 4.3.5

There is a formula you can use as well! The solution to

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx\right]}$$

where

$$\mu(x) = e^{\int P(x)\,dx}$$

EXAMPLE 4.3.6

Solve $x\dfrac{dy}{dx} + 2xe^x y = xe^x$.

Solution. First, divide by x:

$$\frac{dy}{dx} + 2e^x y = e^x$$

$$\mu(x) = e^{\int 2e^x\,dx} = e^{2e^x}$$

Now, using the formula:

$$\begin{aligned}
y &= \frac{1}{\mu(x)} \int \mu(x) Q(x)\, dx \\
&= \frac{1}{e^{2e^x}} \left[\int e^{2e^x} e^x \, dx \right] \\
&= \frac{1}{e^{2e^x}} \left(e^{2u} \right) \\
&= \frac{1}{e^{2e^x}} \left(\frac{e^{2u}}{2} + C \right) \\
&= \frac{1}{e^{2e^x}} \left(\frac{e^{2e^x}}{2} + C \right) \\
&= \frac{1}{2} + \frac{C}{e^{2e^x}}
\end{aligned}$$

There is an important result regarding these ODEs:

THEOREM 4.3.7

Assume P and Q are continuous functions on an interval I. Then, for each $x_0 \in I$ and any $y_o \in \mathbb{R}$, the IVP

$$y' + P(x)y = Q(x)$$

with $y(x_0) = y_0$ has exactly one solution on I.

REMARK 4.3.8

This is not true for other IVPs, some have no solutions, and some have more than one (some even have ∞-many!).

4.7 Newton's Law of Cooling

The law states that an object's temperature changes at a rate that is proportional to the difference between the temperature of the object and the ambient temperature; that is, the temperature of the room, T_{room}. The formula is:

$$\frac{dT}{dt} = -K\left(T - T_{\text{room}}\right)$$

where

- ${}^{dT}/_{dt}$ = rate of change of temperature
- $-K$ = a constant
- T = temperature at time t
- T_{room} = temperature of surroundings (constant)

Q: Why is the constant negative?

A: If $T > T_{\text{room}}$, we would expect the object to be cooling, so ${}^{dT}/_{dt} < 0$. This means the coefficient needs to be negative since $T - T_{\text{room}} > 0$. On the other hand, if $T < T_{\text{room}}$, then the object is heating up, so ${}^{dT}/_{dt} > 0$, but $T - T_{\text{room}} < 0$, so again we need a negative constant.

EXERCISE 4.7.1

The solution to this separable and linear ODE is $T(t) = Ce^{-Kt} + T_{\text{room}}$ for $C, K \in \mathbb{R}$. Also, K can be determined with extra info.

Notice that $\lim_{t\to\infty} T(t) = T_{\text{room}}$, as expected.

EXAMPLE 4.7.2

For $\dfrac{dT}{dt} = -K(T-25)$, if the object was initially at $0°C$, and after 10 minutes it was at $5°C$, solve the ODE.
Here, T is in $°C$ and t is in minutes.
Solution. We know $T = Ce^{-Kt} + 25$, and $T(0) = 0$, $T(10) = 5$. First, $T(0) = 0 \implies 0 = C + 25$, so $C = -25$. Therefore,

$$T = -25e^{-Kt} + 25$$

Next,

$$\begin{aligned} &T(10) = 5 \\ \implies &\frac{-20}{-25} = e^{-10K} \\ \implies &\frac{4}{5} = e^{-10K} \\ \implies &K = -\frac{1}{10}\ln\left(\frac{4}{5}\right) \end{aligned}$$

So,

$$T = -25e^{1/10\ln(4/5)t} + 25$$

4.8 Models of Population Growth

The two models we will examine are:

(I) Natural Growth (Exponential Growth)

(II) Logistic Growth

Natural Growth: It makes sense to assume that population grows at a rate proportional to the size of the population. So, if P = population, and t = time, then

$$\frac{dP}{dt} = kP$$

where k = constant, roughly equal to birth rate − death rate.

It's separable! If $p \not\equiv 0$:

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\begin{aligned} \implies &\ln|P| = kt + C \\ \implies &|P| = e^C e^{kt} \\ \implies &P = \pm e^C e^{kt} \end{aligned}$$

Say $\pm e^C = A \in \mathbb{R}$, then

$$P = Ae^{kt}$$

Note that $P(0) = Ae^0 = A$, so $A =$ initial population. So, the solution to the IVP ${}^{dP}/_{dt} = kP$ with $P(0) = P_0$ is

$$P = P_0 e^{kt}$$

Is natural growth a good model? Say we have 1000 bacteria in a Petri dish, and we observe that 300 new bacteria are formed after 1 hour. It is reasonable to assume that 2000 bacteria would spawn 600 in one hour, isn't it? Well, yes! That is, until they run out of food!

Shouldn't we also take the environment into account? After a certain population, there won't be enough food/space to support any more growth. So it seems like natural growth is fine as long as there are lots of resources (that is, for small populations). But once population nears its limit, it won't be a good model any more.

This limit is called the Carrying Capacity; that is, the maximum population that the environment can support in the long run.

Denote the carrying capacity by M.

DEFINITION 4.8.1

The **logistical differential equation** is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where $k =$ same constant as in natural growth.

REMARK 4.8.2

Sometimes written

$$\frac{dP}{dt} = kP(M - P)$$

but K is different here.

Let's examine some cases:

(I) If $P \ll M$, then $1 - {}^{P}/_{M} \approx 1$, so ${}^{dP}/_{dt} \approx kP$ (natural growth!)

(II) If $P \approx M$, then $1 - {}^{P}/_{M} \approx 0$, so ${}^{dP}/_{dt} \approx 0$, which makes sense as population has reached its limit.

(III) If $P > M$, then ${}^{dP}/_{dt} < 0$, as expected. (The population shrinks as there are not enough resources).

So, this equation appears to be a good model! Let's solve it:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

with $P(0) = P_0$. It's separable!

$$\int \frac{1}{P(1 - {}^{P}/_{M})}\, dP = \int k\, dt = kt + C$$

Use partial fractions! Notice

$$\frac{1}{P(1 - {}^{P}/_{M})} = \frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

So we get

$$\begin{aligned}
&\int \frac{1}{P} + \frac{1}{M-P}\,dP = kt + C \\
&\implies \ln|P| - \ln|M-P| = kt + C \\
&\implies \ln\left|\frac{P}{M-P}\right| = kt + C \\
&\implies \ln\left|\frac{M-P}{P}\right| = -kt - C \\
&\implies \left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt} \\
&\implies \frac{M-P}{P} = \pm e^{-C}e^{-kt} = Ae^{-kt} \qquad\qquad \text{where } A = \pm e^{-C} \\
&\implies \frac{M}{P} - 1 = Ae^{-kt} \\
&\implies P = \frac{M}{1 + Ae^{-kt}}
\end{aligned}$$

Next, use $P(0) = P_0$ to get A:

$$P_0 = \frac{M}{1+A} \implies 1 + A = \frac{M}{P_0} \implies A = \frac{M - P_0}{P_0}$$

So, the solution to $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, $P(0) = P_0$ is

$$\boxed{P = \frac{M}{1 + Ae^{-kt}}}$$

where $\boxed{A = \frac{M - P_0}{P_0}}$.

EXAMPLE 4.8.3

Scientists took 100 wolves and let them go in a walled-off nature preserve. They estimated the carrying capacity to be 1500, and after one year, there were 150 wolves.
Assuming logistic growth, find $P(t)$.
Solution.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{1500}\right)$$

$P(0) = 100$, and $P(1) = 150$. So,

$$P = \frac{1500}{1 + Ae^{-kt}}$$

with $A = {}^{(1500-100)}/_{100} = 14$. So,

$$P = \frac{1500}{1 + 14e^{-kt}}$$

Lastly, use $P(1) = 150$ to get k.

$$150 = \frac{1500}{1 + 14e^{-k}} \implies 1 + 14e^{-k} = 10 \implies e^{-k} = \frac{9}{14} \implies -k = \ln\left(\frac{9}{14}\right)$$

So,

$$P = \frac{1500}{1 + 14e^{\ln(9/14)t}} = \frac{1500}{1 + 14\,(9/14)^t}$$

Q: How long until there are 1000 wolves?
A: Find t:

$$1000 = \frac{1500}{1+14\left(9/14\right)^t} \implies \frac{3}{2} = 1 + 14\left(\frac{9}{14}\right)^t \implies \frac{1}{28} = \left(\frac{9}{14}\right)^t \implies t = \frac{\ln\left(1/28\right)}{\ln\left(9/14\right)} \approx 754 \text{ years}$$

For Fun: There are other models too:

(I) Taking harvesting/hunting into account:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - C$$

where C is the harvesting constant.

(II) If a population is **too sparse**, they may go extinct:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{N}{P}\right)$$

where $N =$ minimum population to prevent extinction.

Chapter 5

Numerical Series

5.1 Introduction to Series

DEFINITION 5.1.1: Infinite series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$$

This is a formal expression since we don't know what this means numerically.

EXAMPLE 5.1.2: Infinite series

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

(iii) $\displaystyle\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots$

Overall Goal: Determine of a given series of numbers **converges** or **diverges**.

Wait a minute! What do these words mean for an infinite sum?!

Well, we need to somehow convert a series to a sequence, which we know how to take a limit of!

DEFINITION 5.1.3: Sequence of Partial Sums

If $\sum\limits_{n=1}^{\infty} a_n$ is a series, define its **sequence of partial sums**, $\{S_n\}$, as $S_n = a_1 + a_2 + \cdots + a_n$

EXAMPLE 5.1.4

For $\sum\limits_{n=1}^{\infty} 1/n$, $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, etc.

Now, we can define convergence/divergence.

DEFINITION 5.1.5: Convergence, Divergence

A series $\sum_{n=1}^{\infty} a_n$ **converges** to $S \in \mathbb{R}$ if $\lim_{n\to\infty} S_n = S$. Here S is called the **sum** of the series. If $\{S_n\}$ diverges, we say the series **diverges**.

EXAMPLE 5.1.6

Using partial sums, determine if $\sum_{n=0}^{\infty}(-1)^n$ converges or diverges.

Solution. Partial sums:

- $S_1 = 1$
- $S_2 = 0$
- $S_3 = 1$
- $S_4 = 0$
- etc.

Clearly, $\lim_{n\to\infty} S_n$ does not exist, so the series diverges.

EXAMPLE 5.1.7

Using partial sums, determine if $\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right]$ converges or diverges.

Solution. Partial sums:

- $S_1 = 1 - 1/2$
- $S_2 = 1 - 1/2 + 1/2 - 1/3 = 1 - 1/3$
- $S_3 = 1 - 1/2 + 1/2 - 1/3 - 1/3 - 1/4 = 1 - 1/4$

There's a pattern! $S_n = 1 - \frac{1}{n+1}$, so $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right] = 1$. Therefore, the series converges to 1.

REMARK 5.1.8

The above series is called a **Telescoping series**.

Let's examine a famous series: The **Harmonic series**.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Does it converge or diverge? Say it converges to S, so

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1}{n} \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \cdots \\
&= S
\end{aligned}$$

So $S > S$, a contradiction. Thus, the series diverges.

REMARK 5.1.9

There are many proofs that the Harmonic series diverges, but this is my favourite. First published in 1976!

It turns out that finding sums of series is hard in general. The partial sum method rarely works, but it does work for telescoping series, as we saw.

It also works for *geometric series*, let's explore these now!

5.2 Geometric Series

DEFINITION 5.2.1: Geometric Series

A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots$$

for some $r \in \mathbb{R}$.

Let's figure out where the series converges!

Case 1: $r = 1$. Then the series is $\sum\limits_{n=0}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$. So $S_n = n$, and $\lim\limits_{n\to\infty} S_n = \infty$, so the series diverges in this case.

Case 2: $r = -1$. Then the series is $\sum\limits_{n=0}^{\infty} (-1)^n$ which we know diverges from before.

Case 3: $r \neq 1$. Then

$$S_n = 1 + r + r^2 + \cdots + r^n$$

and

$$rS_n = r + r^2 + \cdots + r^{n+1}$$

So, $S_n - rS_n = 1 - r^{n+1}$, therefore

$$S_n = \frac{1 - r^{n+1}}{1 - r}$$

Thus, $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{1 - r^{n+1}}{1 - r} = \left(\dfrac{1}{1 - r}\right) \lim\limits_{n\to\infty} [1 - r^{n+1}]$.

Clearly, $\lim\limits_{n\to\infty} [1 - r^{n+1}]$ diverges if $|r| > 1$, but if $|r| < 1$ then $\lim\limits_{n\to\infty} [1 - r^{n+1}] = 1$. Thus,

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}}$$

if $\boxed{|r| < 1}$ and diverges otherwise.

5.3 Arithmetic of Series

Before we explore geometric series further, let's look at some arithmetic properties of series.

THEOREM 5.3.1

Suppose $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$ *and* $k \in \mathbb{R}$.

(1) $\sum_{n=1}^{\infty} ka_n = kA$

(2) $\sum_{n=1}^{\infty} a_n \pm b_n = A \pm B$

Proof of 5.3.1

Follows from limit properties.

Also, if we know something about the tail of a series, then we can draw conclusions about the whole series!

THEOREM 5.3.2

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\sum_{n=j}^{\infty} a_n$ *also converges for each* $j \geqslant 1$.

If $\sum_{n=j}^{\infty} a_n$ *converges for some* j*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof of 5.3.2

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{j-1} + \sum_{n=j}^{\infty} a_n$$

and $a_1 + a_2 + \cdots + a_{j-1} \in \mathbb{R}$, does not affect convergence. So, convergence only depends on the tail! Changing finitely many terms will not affect convergence.

EXAMPLE 5.2.3

Determine whether the following series is convergent or divergent. If a series is convergent, find its sum.

(i) $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

Solution. Converges to $\dfrac{1}{1 - {}^{2}/_{3}} = 3$.

(ii) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution. Converges to

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \left(\frac{1}{1 - {}^{1}/_{2}}\right) = 1.$$

(iii) $\sum_{n=0}^{\infty} 3 \left(\frac{3}{2}\right)^n$

Solution. Diverges since $|r| = {}^{3}/_{2} > 1$.

(iv) $\sum_{n=0}^{\infty} 2^{3n} 3^{-2n}$

Solution. Converges to

$$\sum_{n=0}^{\infty} 2^{3n}3^{-2n} = \sum_{n=0}^{\infty} \frac{8^n}{9^n} = \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n = \frac{1}{1 - {}^8\!/\!{}_9} = 9.$$

(v) $\displaystyle\sum_{n=1}^{\infty} (25)8^n 5^{-n}$

Solution. Diverges since $|r| = {}^8\!/\!{}_5 > 1$.

Interesting Application: Decimals to Fractions

We can use geometric series to write an infinite repeating decimal as a fraction!

EXAMPLE 5.2.4

Use the geometric series to write the decimal $3.2131313\cdots = 3.2\overline{13}$ as a fraction (it doesn't need to be in lowest terms).

Solution.

$$\begin{aligned} 3.2131313\cdots &= 3.2 + \frac{13}{10^3} + \frac{13}{10^5} + \frac{13}{10^7} + \cdots \\ &= \frac{32}{10} + \frac{13}{10^3}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) \\ &= \frac{32}{10} + \frac{13}{10^3}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n \\ &= \frac{32}{10} + \frac{13}{1000}\left(\frac{1}{1 - {}^1\!/\!{}_{100}}\right) \\ &= \frac{3181}{900} \end{aligned}$$

A Real-World Application

Suppose a spaceship is firing a laser beam at a planet with two layers of shields. These shields reflect one-third of the beam, absorb five-ninths of the beam, and transmit one-ninth of the beam. If the beam has initial intensity I, what fraction is transmitted to the other side?

Solution. The total that gets through is:

$$\frac{I}{81} + \frac{I}{9(81)} + \frac{I}{9^2(81)} + \cdots = \frac{I}{81}\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n = \frac{I}{81}\left(\frac{1}{1 - {}^1\!/\!{}_9}\right) = \frac{I}{81}\left(\frac{9}{8}\right) = \frac{I}{72}$$

Finding sums of series, in general, is hard. While we can do it for geometric series and telescoping series, we can't usually get a nice formula for S_n, the partial sums

Soon, we will prove that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, but what is the sum?

First a few partial sums:

- $S_1 = 1$
- $S_2 = 1 + {}^1\!/\!{}_4 = 1.25$
- $S_3 = 1 + {}^1\!/\!{}_4 + {}^1\!/\!{}_9 = 1.36111\ldots$

- $S_4 = 1.4236\ldots$

Sum? 1.75? 2? $\pi^2/6$? Yes!

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

This is difficult to prove though!

So, from now on, we will focus on determining if a series converges or diverges, and not actually finding the sum.

Let's start developing some tests for convergence/divergence.

5.3 Divergence Test

First, let us prove a theorem:

THEOREM 5.3.1

If $\sum\limits_{n=1}^{\infty} a_n$ *converges, then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof of 5.3.1

Suppose $\sum\limits_{n=1}^{\infty} a_n$ converges, say $\sum\limits_{n=1}^{\infty} a_n = S$.
Let $\{S_n\}$ be a sequence of partial sums, so $S_n = a_1 + a_2 + \cdots + a_n$ and $\lim\limits_{n\to\infty} S_n = S$. By sequence of limit properties we get $\lim\limits_{n\to\infty} S_{n-1} = S$ too, and $S_n - S_{n-1} = a_n$. Thus, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} S_n - S_{n-1} = S - S = 0$.

The Divergence Test is the contrapositive of the above theorem:

THEOREM 5.3.2: Divergence Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ *(or DNE), then* $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

REMARK 5.3.3

This test tells only ever tells us a series **diverges**, never that a series converges. So, if you are checking $\sum\limits_{n=1}^{\infty} a_n$, if $\lim\limits_{n\to\infty} a_n$

- $= 0$, get no info;
- $\neq 0$, series diverges.

Q: Can you think of a series such that $\sum\limits_{n=1}^{\infty} a_n$ diverges, even though $\lim\limits_{n\to\infty} a_n = 0$?

A: Yes! The Harmonic Series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.

So, be careful!

When to use the Divergence Test: First! It is a good idea to see if the test works before moving on to more complicated tests!

EXAMPLE 5.3.4

Use Divergence Test to draw a conclusion for the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

Solution.
$\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, so the series diverges by the Divergence Test.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{2^n}$

Solution. $\displaystyle\lim_{n\to\infty} \frac{2^n+3^n}{2^n} = \lim_{n\to\infty} \left[1 + \left(\frac{3}{2}\right)^n\right] = \infty \neq 0$, so the series diverges by the Divergence Test.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution. $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$, so the Divergence Test fails!

(iv) $\displaystyle\sum_{n=1}^{\infty} \arctan(n)$

Solution. $\displaystyle\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$, so the series diverges by the Divergence Test.

5.5 Tests for Positive Series

DEFINITION 5.5.1: Positive

A series $\sum_{n=1}^{\infty} a_n$ is called **positive** if $a_n \geqslant 0$ for all n.

There are a few tests that only work on positive series:

(I) Integral Test

(II) Comparison Test

(III) Limit Comparison Test

Let's examine these now!

The Integral Test

THEOREM 5.5.2: Integral Test

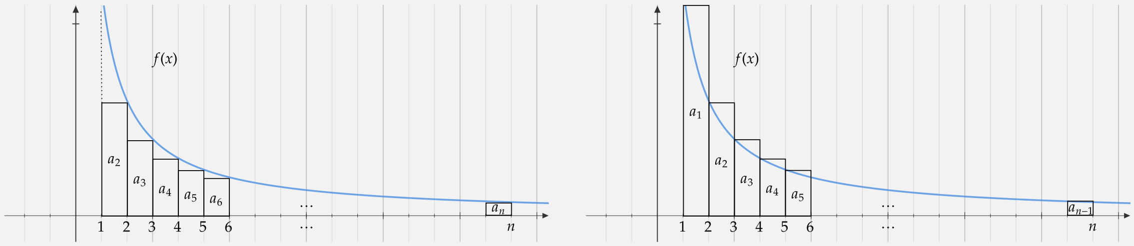
Suppose $f(x)$ is continuous, positive, and decreasing for $x \in [1, \infty)$. Let $a_n = f(n)$. Then, $\sum_{n=1}^{\infty}$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

REMARK 5.5.3

These three conditions don't need to hold for all $x \geqslant 1$, just **eventually** (for $x \geqslant m$ for some $m \in \mathbb{R}^+$), which is the case most of the time when dealing with series and improper integrals!

Proof of 5.5.2

We can cleverly look at two different approximations for the area under $f(x)$ (which is continuous, positive, and decreasing). $y = f(x)$, $f(n) = a_n$:



($\star$) First figure:

$$a_2 + a_3 + a_4 + \cdots + a_n \leqslant \int_1^n f(x)\,dx$$

($\star\star$) Second figure:

$$a_1 + a_2 + \cdots + a_{n-1} \geqslant \int_1^n f(x)\,dx$$

($\Longleftarrow$) Suppose $\int_1^\infty f(x)\,dx$ converges. ($\star$) says

$$\sum_{k=2}^{n} a_k \leqslant \int_1^n f(x)\,dx \leqslant \int_1^\infty f(x)\,dx$$

since $f(x) \geqslant 0$. So,

$$S_n = a_1 + \sum_{k=2}^{\infty} a_k \leqslant a_1 + \int_1^\infty f(x)\,dx$$

which is a constant, say M. So, $0 \leqslant S_n \leqslant M$ for all n. Also, $S_{n+1} = S_n + a_{n+1} \geqslant S_n$ since $a_{n+1} > 0$. So $\{S_n\}$ is a bounded monotonic sequence! Therefore, $\{S_n\}$ converges by MCT, which means $\sum\limits_{n=1}^{\infty} a_n$ converges.

($\Longrightarrow$) If $\int_1^\infty f(x)\,dx$ diverges, then

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$$

and ($\star\star$) says

$$\int_1^n f(x)\,dx \leqslant \sum_{k=1}^{n-1} a_k = S_{n-1}$$

so $\lim\limits_{n\to\infty} S_{n-1} = \infty$, which means $\sum\limits_{n=1}^{\infty} a_n$ diverges.

REMARK 5.5.4

When to use the Integral Test!

- When the series "looks like" it can be integrated; that is, when there are terms like $\ln(n)$, e^n, etc.
- Compared to other tests, the Integral Test takes longer to use, it is better used as a last resort!

REMARK 5.5.5

Don't forget to show the function is *continuous*, *positive*, and *decreasing*!

EXAMPLE 5.5.6

Determine whether the following series is convergent or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution. First, $f(x) = \dfrac{1}{x^2}$ is continuous, positive, and decreasing for $x \geqslant 1$. So we can apply the Integral Test. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx$ converges (p-integral with $p = 2 > 1$), so the series converges.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Solution. Does the Divergence Test work?

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \ldots \text{ no info}$$

Let $f(x) = \dfrac{\ln(x)}{x}$, f is clearly continuous and positive for $x > 1$, but is it decreasing? Let's check!

$$f'(x) = \frac{1 - \ln(x)}{x^2} = 0$$

Before $x = e$, f' is positive and increasing.
After $x = e$, f' is negative and f is decreasing.
So, it is decreasing eventually (for $x \geqslant e$). So, we can use the Integral Test!

$$\int_1^{\infty} \frac{\ln(x)}{x}\, dx = \lim_{b\to\infty} \left[\frac{(\ln(x))^2}{2}\right]_1^b = \lim_{b\to\infty} \frac{(\ln(b))^2}{2} = \infty$$

So, the series diverges.

Q: For which $p \in \mathbb{R}$ is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

A:

Proof of 5.5.7

Case 1: $p < 0$. $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = \infty$, therefore the series diverges by the Divergence Test.

Case 2: $p = 0$. $\displaystyle\lim_{n\to\infty} \frac{1}{n^0} = 1$, therefore the series diverges by the Divergence Test.

Case 3: $p > 0$. $f(x) = \dfrac{1}{x^p}$ is continuous, positive, and decreasing for $x \geqslant 1$. Also, we know $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$ converges if and only if $p > 1$.

THEOREM 5.5.7: p-Series Test

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$, *and diverges if* $p \leqslant 1$.

EXAMPLE 5.5.8: p-Series

Determine whether the following series is convergent or divergent.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$
  Solution. p-series with $p = {}^3\!/\!_2 > 1$, therefore the series converges.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
  Solution. p-series with $p = {}^1\!/\!_2 < 1$, therefore the series diverges.

REMARK 5.5.9

Note that the series does *not* converge to what the integral converges to! For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

but $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. However, we *can* use the integral test to approximate the sum of a series!

DEFINITION 5.5.10: Remainder

The **remainder** is the error in using S_n to approximate $\sum\limits_{n=1}^{\infty} a_n = S$, so

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$$

If $a_n = f(n)$ and $f(x)$ is continuous, positive, and decreasing, we know that

$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_n^{\infty} f(x)\,dx}$$

from the proof of the Integral Test.

So we get an upper bound on the remainder!

EXAMPLE 5.5.11

Find an upper bound on the error if we use S_{10} to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution. $R_{10} \leqslant \displaystyle\int_{10}^{\infty} \frac{1}{x^2}\,dx = \text{exercise} = {}^1\!/\!_{10}$, so the error is at most 0.1.

EXAMPLE 5.5.12

How many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ with an error of at most 0.001?

Solution. Need $R_n \leqslant 0.001$, but we know $R_n \leqslant \displaystyle\int_n^{\infty} \frac{1}{x^2}\,dx$, so solve

$$\int_n^{\infty} \frac{1}{x^2}\,dx \leqslant 0.001 \implies \frac{1}{n} \leqslant 0.001 \implies n \geqslant 1000$$

We can improve our estimate, rather than just using S_n:

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n = S - S_n \leqslant \int_n^{\infty} f(x)\,dx$$

so

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant S \leqslant S_n + \int_n^{\infty} f(x)\,dx$$

Back to the $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ example, using 10 terms:

$$S_{10} + \int_{11}^{\infty} \frac{1}{x^2}\,dx \leqslant S \leqslant S_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,dx \implies S_{10} + \frac{1}{11} \leqslant S \leqslant S_{10} + \frac{1}{10} \implies 1.640677 \leqslant S \leqslant 1.649768$$

Take midpoint: $S \approx 1.64522$. The error is actually $\leqslant 0.0003$.

The Comparison Test

Just like improper integrals, we can compare series!

THEOREM 5.5.13: Comparison Test

Assume $0 \leqslant a_n \leqslant b_n$ *for* $n \in \mathbb{N}$ *(or eventually)*

(1) If $\sum\limits_{n=1}^{\infty} b_n$ *converges, then* $\sum\limits_{n=1}^{\infty} a_n$ *converges too.*

(2) If $\sum\limits_{n=1}^{\infty} a_n$ *diverges, then* $\sum\limits_{n=1}^{\infty} b_n$ *diverges too.*

Proof of 5.5.13

Notice that 2 is the contrapositive 1, so let's prove 2.

Let $\{S_n^a\}$ and $\{S_n^b\}$ be the partial sum sequences for $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$, respectively.

Suppose $\sum\limits_{n=1}^{\infty} a_n$ diverges. Then, since $a_n \geqslant 0$ for all n, $\lim\limits_{n\to\infty} S_n^a = \infty$. But $S_n^a \leqslant S_n^b$ for all n, so $\lim\limits_{n\to\infty} S_n^b = \infty$ too, which means $\sum\limits_{n=1}^{\infty} b_n$ diverges.

EXAMPLE 5.5.14

Determine whether the following series is convergent or divergent.

(i) $\sum\limits_{n=1}^{\infty} \frac{n}{n^3+7}$

Solution. Note that $0 \leqslant \frac{n}{n^3+7} \leqslant \frac{n}{n^3} = \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). So, the given series converges by comparison.

(ii) $\sum\limits_{n=2}^{\infty} \frac{n+7}{n^2-1}$

Solution. Note that $\frac{n+7}{n^2-1} \geqslant \frac{n}{n^2-1} \geqslant \frac{n}{n^2} = \frac{1}{n} \geqslant 0$ and $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges (Harmonic Series). So, the given series diverges by comparison.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - n}{n^4 + 7}$

Solution. Note that $\dfrac{n^3 - n}{n^4 + 7} \leqslant \dfrac{n^3}{n^4 + 7} \leqslant \dfrac{n^3}{n^4} = \dfrac{1}{n}$, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so comparison fails.

But it "looks like" $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges! The inequalities were just pointing the wrong way! If only there was a test that didn't require us to use inequalities but still allowed us to compare two series...Well good news!

The Limit Comparison Test (LCT)

THEOREM 5.5.15: LCT

If $0 \leqslant a_n$ and $0 < b_n$ for $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

with $L \neq 0$ and $L < \infty$, then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge.

Proof of 5.5.15

Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$. Then, we can find positive numbers m and M so that $m < L < M$. But since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, if n is large enough we get $m < \dfrac{a_n}{b_n} < M \to \underbrace{\boxed{mb_n < a_n < Mb_n}}_{(\star)}$. So, if one series converges/diverges we can use $(\star)$ and comparison to show the other converges/diverges too.

Cool! We don't need to worry about inequalities!

REMARK 5.5.16

When to use LCT:

- Series like $\displaystyle\sum \frac{\text{powers of } n}{\text{powers of } n}$
- "Almost" geometric series

Strategy: Pick the dominant terms (as $n \to \infty$) in the numerator and denominator to compare to.

EXAMPLE 5.5.17

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - n}{n^4 + 7}$

Solution. Use LCT with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\left(\dfrac{n^3 - n}{n^4 + 7}\right)}{\left(\dfrac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^4 - n^2}{n^4 + 7} = 1$$

so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic Series), so does the given series.

(ii) $\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n + n}$

Solution. This looks like an "almost" geometric series. Use LCT with $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$:

$$\lim_{n\to\infty} \frac{\left(\frac{2^n - 1}{3^n + n}\right)}{\left(\frac{2^n}{3^n}\right)} = \lim_{n\to\infty} \frac{6^n - 3^n}{6^n + n2^n} = \lim_{n\to\infty} \frac{1 - \frac{1}{2^n}}{1 + \frac{n}{3^n}} = 1$$

so since $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges geometric series ($|r| = {}^2\!/_3 < 1$), so does the given series.

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 5n} + 3}{n^{7/4} + 3n - 1}$

Solution. Use LCT with $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$:

$$\lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^2 + 5n} + 3}{n^{7/4} + 3n - 1}\right)}{\left(\frac{1}{n^{3/4}}\right)} = \text{exercise} = 1$$

so since $\sum_{n=1}^{\infty} {}^1\!/_{n^{3/4}}$ diverges (p-series with $p = {}^3\!/_4 \leqslant 1$), so does the given series.

We can also extend the LCT to discuss what happens if $L = 0$ or $L = \infty$:

THEOREM 5.5.18: LCT

If $a_n \geqslant 0$ and $b_n > 0$ for $n \in \mathbb{N}$ (or eventually) and $\lim_{n\to\infty} {}^{a_n}\!/_{b_n} = L$, then:

(1) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

(2) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

Proofs of (2) and (3) are similar to (1), exercises.

REMARK 5.5.19

Make sure if you use (2) or (3), you draw the correct conclusion.

EXAMPLE 5.5.20

Determine if $\sum_{n=2}^{\infty} \frac{[\ln(n)]^3}{\sqrt{n}}$ is convergent or divergent.

Solution. LCT with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\left(\frac{(\ln(n))^3}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} (\ln(n))^3 = \infty,$$

so since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = 1/2 \leqslant 1$), so does the given series.

5.7 Alternating Series

So far, we have only developed tests for positive series. Before we look at how to extend them to all series, let's examine *alternating series*.

DEFINITION 5.7.1: Alternating series

A series is **alternating** if its terms are alternately positive or negative.

EXAMPLE 5.7.2: Alternating series

(i) $1 - 1/2 + 1/3 - 1/4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is alternating.

(ii) $1 - 1/2 - 1/3 + 1/4 + \frac{1}{5} - \cdots$ is *not* alternating.

REMARK 5.7.3

Look for $(-1)^n$, $\cos(n\pi)$, etc.

THEOREM 5.7.4: Alternating Series Test (AST)

Suppose $a_n > 0$ for all n. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. If:

(1) $\{a_n\}$ is non-increasing (eventually): $a_n \geqslant a_{n+1}$ and

(2) $\lim_{n\to\infty} a_n = 0$

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof of 5.7.4

We will prove $\{S_n\}$ converges by proving $\{S_{2n}\}$ and $\{S_{2n+1}\}$ both converge to the same limit. Suppose $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$.

<u>Even Partial Sums</u>:

- $S_2 = a_1 - a_2 > 0$
- $S_4 = \underbrace{(a_1 - a_2)}_{>0} + \underbrace{(a_3 - a_4)}_{>0} > S_2$
- $S_6 = S_4 + \underbrace{(a_5 - a_6)}_{>0} > S_4$
- etc.

In general,

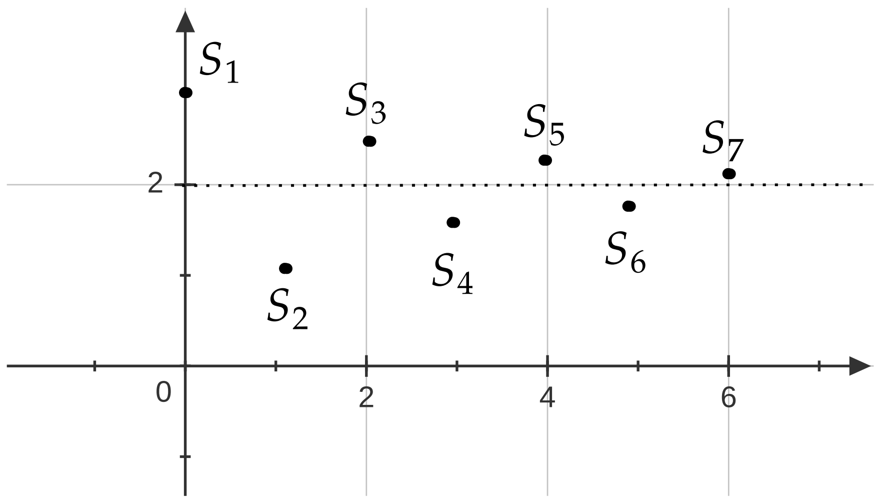
$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2}$$

so $0 < S_2 < S_4 < \cdots < S_{2n} < \cdots$, so $\{S_{2n}\}$ is increasing. Also,

$$S_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{>0} - \underbrace{(a_4 - a_5)}_{>0} - \cdots - \underbrace{(a_{2n-2} - a_{2n-1})}_{>0} - \underbrace{a_{2n}}_{>0}$$

so $S_{2n} \leqslant a_1$. Since $\{S_{2n}\}$ is increasing and bounded above, it converges by MCT. Say $\lim\limits_{n\to\infty} S_{2n} = S$.
Odd Partial Sums: $\lim\limits_{n\to\infty} S_{2n+1} = \lim\limits_{n\to\infty} S_{2n} + a_{2n+1} = S + 0 = S$. So $\lim\limits_{n\to\infty} S_{2n} = S = \lim\limits_{n\to\infty} S_{2n+1}$, which means $\lim\limits_{n\to\infty} S_{2n} = S$. Therefore, the series converges.

Picture of what's going on:



EXAMPLE 5.7.5

Determine whether the following series is convergent or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (*Alternating Harmonic Series*)

Solution. First, $\dfrac{1}{n+1} < \dfrac{1}{n}$, so $\left\{\dfrac{1}{n}\right\}$ is decreasing. Also, $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. So, the Alternating Harmonic Series converges by AST.

(ii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n}}{n\ln(n)}$

Solution. First, $\dfrac{1}{(n+1)\ln(n+1)} \leqslant \dfrac{1}{n\ln(n)}$, so $\left\{\dfrac{1}{n\ln(n)}\right\}$ is decreasing. Also, $\lim\limits_{n\to\infty} \dfrac{1}{n\ln(n)} = 0$. So, the series converges by AST.

Q: What if we are dealing with alternating series, but it doesn't satisfy the hypotheses of the AST?! It's our only test!

A: If an alternating series fails the AST, it means you forgot to check the Divergence Test!

See the following example.

EXAMPLE 5.7.6: You forgot to check the Divergence Test!

Is $\sum_{n=1}^{\infty} \frac{(-1)^n e^{2n}}{e^{2n}+1}$ convergent or divergent?

Solution. Using the Divergence Test:

$$\lim_{n\to\infty} \frac{(-1)^n e^{2n}}{e^{2n}+1} = \lim_{n\to\infty} \frac{(-1)^n}{1+{}^1/e^{2n}}$$

does not exist. Thus, the series diverges.

Estimating Sums of Alternating Series

Suppose we have an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ that converges by AST.

We know from the proof that the odd partial sums approach the actual sum from above, while the even partial sums approach from below.

This means the actual sum lies between any two consecutive partial sums, so the error satisfies

$$|R_n| = |S - S_n| \leqslant |S_{n+1} - S_n| = |\pm a_{n+1}| = a_{n+1}$$

which is the next term! So, $\boxed{|R_n| \leqslant a_{n+1}}$.

EXAMPLE 5.7.7

Find an upper bound on the remainder if we use S_{10} to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Solution. $|R_n| \leqslant a_{n+1}$, so $|R_{10}| \leqslant a_{11} = {}^1/{}_{11}$.

EXAMPLE 5.7.8

How many terms are needed to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n n!}$ with an error of at most 0.000005?

Solution. The series converges by AST (exercise)

$$|R_n| \leqslant a_{n+1} = \frac{1}{10^{n+1}(n+1)!},$$

so we want $a_{n+1} \leqslant 0.000005$. Guess and check (since factorials don't have inverses).

- $n = 3$: $\frac{1}{10^4(4)!} \approx 0.000004$ (good!)
- $n = 2$: $\frac{1}{10^3(3)!} \approx 0.0016$ (bad!)

So $n = 3$ works (3 terms).

Q: Is the 121st partial sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{(n)4^n}$ an overestimate or underestimate of the actual sum?

A: First term is positive, so the odd partial sums are above the sum while the even partial sums are below. So S_{121} is an *overestimate*!

REMARK 5.7.9

If the first term is negative, then the odd partial sums are underestimates while the even partial sums are overestimates.

5.8 Absolute versus Conditional Convergence

So far we've examined tests for positive series, and a test for alternating series, but what can we do about the series that have non-alternating assortment of positive and negative terms? Is there a way to make our tests for positive series work in this case too? Yes! Just like for improper integrals, we use absolute values and discuss absolute convergence!

DEFINITION 5.8.1: Absolutely convergent

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

REMARK 5.8.2

Note that if a series only has positive terms then absolute convergence is the same as convergence because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$.

EXAMPLE 5.8.3

(i) Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely convergent?

Solution. Yes, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (p-series, $p = 2 > 1$)

(ii) Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ absolutely convergent?

Solution. No, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (Harmonic Series)

REMARK 5.8.4

Don't say "absolutely divergent," that makes it sound worse than it is!

We know $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by AST, and we have a name for the series that behave this way:

DEFINITION 5.8.5: Conditionally convergent

A series is **conditionally convergent** if it is convergent, but not absolutely convergent.

We also have an analogue of the ACT for series:

THEOREM 5.8.6: ACT

If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

REMARK 5.8.7

Note that unless $a_n \geqslant 0$ for all n, $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=1}^{\infty} a_n$ will converge to different values!

Proof of 5.8.6

(Similar to the proof of ACT for integrals). Suppose $\sum_{n=1}^{\infty}|a_n|$ converges. Note that $0 \leqslant a_n + |a_n| \leqslant 2|a_n|$. Since $\sum_{n=1}^{\infty}|a_n|$ converges, $\sum_{n=1}^{\infty} 2|a_n|$ converges, so by comparison $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges, too. But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}[(a_n + |a_n|) - |a_n|] = \underbrace{\sum_{n=1}^{\infty}(a_n + |a_n|)}_{\text{converges}} - \underbrace{\sum_{n=1}^{\infty}|a_n|}_{\text{converges}}$$

So $\sum_{n=1}^{\infty} a_n$ converges.

Cool! So, to prove a series converges we can prove it is absolutely convergent instead, which allows us to use tests like Integral/Comparison/Limit comparison!

EXAMPLE 5.8.8

Is $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n^3)}{n^3}$ convergent?

Solution. Let's check absolute convergence! $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin(n^3)}{n^3}\right|$: We know $0 \leqslant \left|\frac{\sin(n^3)}{n^3}\right| \leqslant \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges (p-series, $p = 3 > 1$), so $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin(n^3)}{n^3}\right|$ converges by comparison. So, the given series converges by ACT.

A typical question will ask "are the following series absolutely convergent, conditionally convergent, or divergent," then you should:

- Step 1: Try the Divergence Test
- Step 2: Check absolute convergence with old tests for positive series.
- Step 3: Check conditional convergence with AST.

REMARK 5.8.9

Doing Step 2 before Step 3 is a good idea since if the series converges absolutely you're done, while if it converges by AST you still need to check absolute convergence.

EXAMPLE 5.8.10

Do the following series converge absolutely, conditionally, or diverge?

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{3^n}$

Solution. Divergence Test: $\displaystyle\lim_{n\to\infty}\frac{(-1)^n 4^n}{3^n}$ does not exist, so the series diverges.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n^2+n}}{n^{3/2}}$

Solution. Note the Divergence Test fails. Check absolute convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n^2+n}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{\sqrt{n^2+n}}{n^{3/2}}$$

LCT with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$:

$$\lim_{n\to\infty} \frac{\left(\dfrac{\sqrt{n^2+n}}{n^{3/2}}\right)}{\left(\dfrac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$$

So since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, so does the given series, so our series is *not* absolutely convergent. Let's check AST: Clearly $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n^{3/2}} = 0$, but is the sequence decreasing? We could use derivatives to check, but instead let's try it directly:

$$\begin{aligned}
\frac{\sqrt{(n+1)^2+(n+1)}}{(n+1)^{3/2}} < \frac{\sqrt{n^2+n}}{n^{3/2}} &\iff n^{3/2}\sqrt{n^2+2n+1+n+1} < \sqrt{n^2+n}(n+1)^{3/2} \\
&\iff n^3(n^2+3n+2) < (n^2+n)(n+1)^3 \\
&\iff n^5+3n^4+2n^3 < (n^2+n)(n^3+3n^2+3n+1) \\
&\iff 2n^3 < 3n^3+n^2+n^4+3n^3+3n^2+n \\
&\iff 0 < n^4+4n^3+4n^2+n
\end{aligned}$$

which is true for all $n \geqslant 1$. So the sequence is decreasing, which means the series converges by AST. Thus, our given series is conditionally convergent.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{\pi^n}$

Solution. Note the Divergence Test fails. Wait a minute! This series only has positive terms! So absolute convergence = convergence! Also, it's a geometric series ($|r| = {}^{e}/_{\pi} < 1$), so the series *converges absolutely*.

Aside on Rearranging The Order of a Series

We have been discussing "sums" of infinite series, but it turns out that these can behave very strangely!

Q: Can we rearrange the order in which we sum the series?

A: Sometimes! If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent with sum S, then any rearrangement of the terms will also have the sum S.

What about conditional convergence? Let's see! Soon, we will prove that the sum of the alternating Harmonic Series is $\ln(2)$.

$(\star)$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2)$$

Divide by 2:

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots = \frac{1}{2}\ln(2)$$

$(\star\star)$ Add in some 0's:

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln(2)$$

$(\star\star\star)$ Computing $(\star) + (\star\star)$:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln(2)$$

But $(\star\star\star)$ is a rearrangement of $(\star)$. So changing the order changes the sum!

In fact, Riemann proved that if $\sum\limits_{n=1}^{\infty} a_n$ is conditionally convergent, then by rearranging we can make it add up to *any* real number (or $\pm\infty$)!

5.9 Ratio and Root Tests

We have two more tests to examine!

THEOREM 5.9.1: Ratio Test

Let $\sum\limits_{n=1}^{\infty} a_n$ be a series, and assume $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, with $L \in \mathbb{R}$ or $L = \infty$.

(1) If $L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $L > 1$ (or $L = \infty$), then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

(3) If $L = 1$, we get no info.

Proof of 5.9.1

Proof of (1) Suppose $L < 1$ (Idea: compare to a geometric series!) Since $L < 1$, we can pick $r \in \mathbb{R}$ with $L < r < 1$. Since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < r$, for large enough n, say $n \geqslant N$, $\left|\frac{a_{n+1}}{a_n}\right| < r$, or $|a_{n+1}| < r|a_n|$. So

$$\begin{aligned}
&|a_{N+1}| < r|a_N| \\
&|a_{N+2}| < r|a_{N+1}| < r^2|a_N| \\
&|a_{N+3}| < \cdots r^3|a_N|
\end{aligned}$$

$(\star)$ In general: $|a_{N+K}| < \cdots < r^k|a_N|$.

Furthermore, $\sum\limits_{n=1}^{\infty} |a_N| r^n$ converges (geometric series, $r < 1$). So, by $(\star)$ and comparison, $\sum\limits_{n=N+1}^{\infty} |a_n|$ converges, but then so does $\sum\limits_{n=1}^{\infty} |a_n|$. Thus, $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.

Proof of (2) Suppose $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or $L = \infty$). Then eventually $\left|\frac{a_{n+1}}{a_n}\right| > 1$, or $|a_{n+1}| > |a_n| > 0$. Hence, the size of the terms is increasing eventually, so $\lim\limits_{n\to\infty} a_n \neq 0$. Therefore, the series diverges by the Divergence Test.

Proof of (3) Consider $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. In both cases $L = 1$, but one converges and the one diverges. So, if $L = 1$ we get no info.

REMARK 5.9.2

When to use:

- Factorials! (see what I did there?!)
- Also works on some "almost" geometric series

If you see a factorial (after simplifying), use the Ratio Test first! Even before the Divergence Test.

EXAMPLE 5.9.3

Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

Solution. Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\left(\frac{n!}{3^n}\right) = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$$

So the given series *converges absolutely*.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 9^n}{n2^n}$

Solution. Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}9^{n+1}}{(n+1)2^{n+1}}\left[\frac{n2^n}{(-1)^n 9^n}\right]\right| = \lim_{n\to\infty}(9)\left(\frac{1}{2}\right)\left(\frac{n}{n+1}\right) = \frac{9}{2} > 1$$

So, the series *diverges*.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution. Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{n!}{n^n}\right)\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e > 1$$

So, the series *diverges*. This shows $n^n \gg n!$ as $n \to \infty$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{5^n}$

Solution. "Almost geometric," Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(n+1)^2+3(n+1)}{5^{n+1}}\left(\frac{5^n}{n^2+3n}\right)\right| = \lim_{n\to\infty}\frac{1}{5}\left(\frac{n^2+2n+1+3n+3}{n^2+3n}\right) = \frac{1}{5} < 1$$

So, the series *converges absolutely*.

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n+1}{3n^4+4}$

Solution. Should use LCT (exercise), but what if we use the Ratio Test?

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{(n+1)^2+2(n+1)+1}{3(n+1)^4+4}\left(\frac{3n^4+4}{n^2+2n+1}\right)\right| \\ &= \lim_{n\to\infty}\left[\frac{(n+1)^2+2(n+1)+1}{n^2+2n+1}\left(\frac{3n^4+4}{3(n+1)^4+4}\right)\right] = 1\end{aligned}$$

Ratio Test fails! We have one more test to examine: the Root Test!

We have one more test to examine: the Root Test!

THEOREM 5.9.4: Root Test

Let $\sum_{n=1}^{\infty} a_n$ *be a series and assume* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in \mathbb{R}$ *or* $L = \infty$.

(1) If $L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(2) If $L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(3) If $L = 1$*, then we get no info.*

The proof is similar to the Ratio Test proof.

REMARK 5.9.5

When to use:

- When all the terms of a series are raised to the power of n

Warning:

- If the Ratio Test fails ($L = 1$), then the Root Test will also fail and vice versa.

EXAMPLE 5.9.6

$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+7}\right)^n$

Solution. Root Test:

$$\lim_{n\to\infty} \left|\left(\frac{n+1}{3n+7}\right)^n\right|^{1/n} = \lim_{n\to\infty} \frac{n+1}{3n+7} = \frac{1}{3} < 1$$

So the series *converges absolutely*.

To help us predict if a series will converge or diverge, let's examine the relative sizes of common functions:

$$\boxed{[\ln(n)]^p \ll n^p \ll x^n \ll n! \ll n^n}$$

for $|x| > 1$.

We have seen most of these already, but let's prove one more:

THEOREM 5.9.7

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Proof of 5.9.7

Using the Ratio Test, we can show that $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for any $x \in \mathbb{R}$. Therefore, by the Divergence Test, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

This shows that $x^n \ll n!$.

Series Test Recap

- Sums of Geometric and Telescoping Series
  - Try to spot these series.

  - If the question says “find the sum,” it’s likely one of these.
- Divergence Test (Any Series)
  - Try this first, unless there is a factorial.
- Integral Test (Positive Series)
  - Last resort when all else fails.
  - Don’t forget continuous, positive, and decreasing.
- p-series $\left(\sum \frac{1}{n^p}\right)$
  - Good for Comparison and Limit Comparison Test.
- Comparison Test (Positive Series)
  - Also a last resort, LCT is usually better.
- LCT (Positive Series)
  - Series of the form $\frac{\text{powers of } n}{\text{powers of } n}$.
  - “Almost” geometric series.
  - Don’t forget: $L = 0$ or $L = 0$ are more complicated!
- Ratio Test (Any Series)
  - Factorials!
  - “Almost” geometric series.
  - $L = 1$ gives no info.
- Root Test (Any Series)
  - When all terms have a power of n.

All the above tests can only discuss absolute convergence or divergence.

- AST (Alternating Series)
  - For proving conditional convergence.

EXERCISE 5.9.8: Series Practice

Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n}}{n!}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n}{n^3-3n+1}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7}{n^3+n^2}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{\sqrt{n}}$.

(v) $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$.

(vi) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{n^3+3}$.

Solutions to 5.9.8:

(i) Ratio Test (absolutely convergent)

(ii) LCT (diverges)

(iii) Divergence Test (diverges)

(iv) LCT or Comparison (diverges)

(v) Root Test (converges absolutely)

(vi) LCT and AST (converges conditionally)

Chapter 6

Power Series

6.1 Introduction to Power Series

DEFINITION 6.1.1: Power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \text{ (centre} = 0)$$

or

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \text{ (centre=}a)$$

where $a_i \in \mathbb{R}$ for all i.

DEFINITION 6.1.2: Domain

The **domain** of a power series is the collection of all $x \in \mathbb{R}$ for which the power series converges.

REMARK 6.1.3

The domain is never empty! The series will always converge (to a_0) at $x =$ centre.

Conventions: To simplify notation, we will use the following conventions in this section for $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$:

(I) When $n = 0$, the term is a_0 for all x, including $x = a$ (so $0^0 = 1$ here!)

(II) If the first few coefficients are zero; that is, $a_0 = a_1 = \cdots = a_k = 0$, then

$$\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=k+1}^{\infty} a_n(x-a)^n$$

In other words, if a coefficient is zero, regardless of what power $(x-a)$ has, that term is zero, and you can discard it.

EXAMPLE 6.1.4

Find the domain of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. Use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{(n+1)!}\right)\left(\frac{n!}{x^n}\right)\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

for all $x \in \mathbb{R}$. So, the series converges for all $x \in \mathbb{R}$, which means the domain is $\mathbb{R}$.

EXAMPLE 6.1.5

Find the domain of $\sum_{n=0}^{\infty}(x-7)^n$.

Solution. Ratio (or Root) Test:

$$\lim_{n\to\infty}\left|\frac{(x-7)^{n+1}}{(x-7)^n}\right| = \lim_{n\to\infty}|x-7| = |x-7|$$

To guarantee the series converges, we need $|x-7| < 1$, or $6 < x < 8$. However, the Ratio Test fails if $|x-7| = 1$; that is, $x = 6$ or $x = 8$, so let's check these separately!

- If $x = 6$: $\sum_{n=0}^{\infty}(6-7)^n = \sum_{n=0}^{\infty}(-1)^n$ diverges.
- If $x = 8$: $\sum_{n=0}^{\infty}(8-7)^n = \sum_{n=0}^{\infty}1^n$ diverges.

So, the domain in this case is $(6, 8)$. In fact, the domain will always be an interval!

THEOREM 6.1.6

For a given power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, there are three possibilities:

(1) The series converges only when $x = a$.
(2) The series converges for all $x \in \mathbb{R}$.
(3) There exists $R \in \mathbb{R}$ such that the series converges absolutely for $|x-a| < R$, diverges if $|x-a| > R$, and may converge or diverge if $|x-a| = R$.

Proof of 6.1.6

For simplicity, let's work with $\sum_{n=0}^{\infty} a_n x^n$ (centre 0), we can shift everything to $x = a$ if needed. We will show that if the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$ and $|x_1| < |x_0|$, then $\sum_{n=0}^{\infty}|a_n x_1^n|$ converges too.

Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, $\lim_{n\to\infty}|a_n x_0^n| = 0$ by the Divergence Test. Therefore, $|a_n x^n| < 1$ eventually.

Next, we can see that

$$|a_n x_1^n| = |a_n x_0^n|\left|\frac{x_1^n}{x_0^n}\right| \leqslant \left|\frac{x_1^n}{x_0^n}\right|$$

eventually. But $\sum_{n=0}^{\infty}\left|\frac{x_1}{x_0}\right|^n$ converges (geometric series $|r| = |x_1/x_0| < 1$), so $\sum_{n=0}^{\infty}|a_n x_1^n|$ converges.

DEFINITION 6.1.7: Radius of convergence

The R in the theorem is called **radius of convergence** of the power series. 6.1.6:

- Case (1) $\implies R = 0$
- Case (2) $\implies R = \infty$
- Case (3) $\implies R \in (0, \infty)$. In this case, the endpoints must be checked separately (without Ratio Test).

DEFINITION 6.1.8: Interval of convergence

The **interval of convergence** is the interval on which the power series converges. So, the interval could be:

- $I = \{a\}$; $R = 0$
- $I = \mathbb{R}$; $R = \infty$
- $I = (a - R, a + R)$; $R \in (0, \infty)$
- $I = [a - R, a + R)$; $R \in (0, \infty)$
- $I = (a - R, a + R]$; $R \in (0, \infty)$
- $I = [a - R, a + R]$; $R \in (0, \infty)$

REMARK 6.1.9

The series converges absolutely on I except maybe at the endpoints.

To find the radius, use the Ratio Test! Note that the Ratio Test limit may not exist! See example 6 in section 6.1. For our assignments and exams it will though.

EXAMPLE 6.1.10

Find the radius and interval of convergence for the following power series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$.

Solution. Ratio Test:

$$\lim_{n\to\infty} \left| \left(\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}} \right) \left(\frac{\sqrt{n}}{3^n (x+4)^n} \right) \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} (3|x+4|) = 3|x+4|$$

We need $3|x+4| < 1$, so $|x+4| < 1/3$. So $R = 1/3$.

The *open* interval (before checking endpoints) is:

$$\left(-4 - \frac{1}{3}, -4 + \frac{1}{3}\right) = \left(-\frac{13}{3}, -\frac{11}{3}\right)$$

<u>Check Endpoints</u>

$x = -\dfrac{13}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \left(-13/3 + 4\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n \left(-1/3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST.

$x = -\dfrac{11}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n \left(-11/3 + 4\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3^n \left(1/3\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series, $p = 1/2 < 1$).

So, the interval of convergence is $[-13/3, -11/3)$.

(ii) $\displaystyle\sum_{n=0}^{\infty} n! x^n$.

Solution. Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So the series diverges unless $x = 0 \implies R = 0$, $I = \{0\}$.

(iii) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^n \ln(n)}$

Solution. Ratio Test:

$$\lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)}\right)\left(\frac{4^n\ln(n)}{(-1)^n x^n}\right)\right| = \lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}\left(\frac{1}{4}\right)|x| = \lim_{n\to\infty}\frac{1/n}{1/(n+1)}\left(\frac{|x|}{4}\right) = \frac{|x|}{4}$$

Need $|x|/4 < 1 \implies |x| < 4$. So, $R = 4$, open interval is $(-4, 4)$.

Check Endpoints

$x = -4$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n(-4)^n}{4^n\ln(n)} = \sum_{n=2}^{\infty}\frac{1}{\ln(n)}$. Note that $\dfrac{1}{\ln(n)} \geqslant \dfrac{1}{n}$ for $n \geqslant 2$, so since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (Harmonic Series), so does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ diverges by comparison.

$x = 4$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n 4^n}{4^n\ln(n)} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)}$ converges by AST.

So, the interval of convergence is $(-4, 4]$.

6.2 Representing Functions as Power Series

A power series, $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a function whose domain is its interval of convergence.

We already know one function as a series: Geometric Series!

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$$

for $|x| < 1$. $R = 1$ and $I = (-1, 1)$.

Let's see what we can say about power series!

THEOREM 6.2.1: Abel's Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I, then f is continuous on I.

Proof of 6.2.1

Beyond the scope of this course.

While this is interesting, soon we will see that we can say a lot more!

We can also use known power series to get power series for other functions. Let's examine the rules first.

Say $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ with radii of convergence R_f and R_g and intervals of convergence I_f and I_g, respectively.

(I) $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-a)^n$. If $R_f \neq R_g$, then the radius of convergence is $R = \min\{R_f, R_g\}$ and the interval is $I_f \cap I_g$. If $R_f = R_g$, then $R > R_f$.

(II) $(x-a)^k f(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n+k}$ where the radius is R_f and the interval is I_f; that is, there is no change.

(III) If $c \in \mathbb{R}$ with $c \neq 0$, and $a = 0$, then $f(cx^k) = \sum_{n=0}^{\infty} a_n c^n x^{nk}$, where we get the radius, R, by solving $|cx^k| < R_f \implies |x| < \sqrt[k]{\frac{R_f}{|c|}}$, so the new radius is $R = \sqrt[k]{\frac{R_f}{|c|}}$. If $R_f = \infty$ then $R = \infty$. The interval is $I = \{x \in \mathbb{R} \mid cx^k \in I_f\}$.

Point is: we can substitute into a known series to form a new one.

EXAMPLE 6.2.2

Find a power series for $f(x) = \dfrac{1}{3-x}$ about $x = 0$.

Solution.

$$\frac{1}{3-x} = \frac{1}{3}\left(\frac{1}{1-x/3}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$$

which is valid for $|x/3| < 1 \implies |x| < 3$ so $R = 3$ and $I = (-3, 3)$.

REMARK 6.2.3

We don't need to check endpoints for geometric series.

EXAMPLE 6.2.4

Find a power series for $f(x) = \dfrac{x^2}{x+7}$ centred at $x = 0$.

Solution.

$$\frac{x^2}{x+7} = \frac{x^2}{7}\left(\frac{1}{1+x/7}\right) = \frac{x^2}{7}\left[\frac{1}{1-(-x/7)}\right] = \frac{x^2}{7}\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{7^{n+1}}$$

for $|-x/7| < 1 \implies |x| < 7$ so $R = 7$ and $I = (-7, 7)$.

EXAMPLE 6.2.5

Find a power series for $f(x) = \dfrac{1}{4-x^2}$ about $x = 0$.

Solution.

$$\frac{1}{4-x^2} = \frac{1}{4}\left(\frac{1}{1-x^2/4}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{4^{n+1}}$$

for $|x^2/4| < 1 \implies |x| < 2$ so $R = 2$ and $I = (-2, 2)$.

What about not centred at $x = 0$?

EXAMPLE 6.2.6

Find a series representation for $f(x) = \dfrac{1}{x}$ centred at $x = 3$.

Solution. The trick is to add and subtract 3.

$$\frac{1}{x} = \frac{1}{(x-3)+3} = \frac{1}{3}\left[\frac{1}{1+\left(\frac{x-3}{3}\right)}\right] = \frac{1}{3}\left[\frac{1}{1-\left(-\frac{(x-3)}{3}\right)}\right] = \frac{1}{3}\sum_{n=0}^{\infty}\left[-\frac{(x-3)}{3}\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{3^{n+1}}$$

for $\left|-\frac{(x-3)}{3}\right| < 1 \implies |x-3| < 3$ so $R = 3$ and $I = (0, 6)$.

6.3 Differentiation and Integration

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, we can differentiate or integrate **term-by-term**:

THEOREM 6.3.1

If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence $R > 0$, then $f(x)$ is differentiable (hence continuous and integrable) on $(a-R, a+R)$, and:

(1) $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$

(2) $\int f(x)\,dx = \sum_{n=0}^{\infty}\left[\frac{a_n(x-a)^{n+1}}{n+1}\right] + C$

Both have radius of convergence R.

REMARK 6.3.2

In (1) always want to change the starting index since if $n = 0$, the term is 0.

REMARK 6.3.3

While the radius doesn't change, the interval *may change*! We need to check the endpoints if we integrate/differentiate.

Proof of 6.3.1

Beyond the scope of this course.

EXAMPLE 6.3.4

Find a power series for $\ln|1+x|$ about $x = 0$.

Solution. We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, so $R = 1$. Then, we get $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$. Integrate:

$$\ln|1+x| = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{n+1}}{n+1}\right] + C$$

First, we can find C by subbing into $x = 0$ (the centre) (since we want a series for $\ln|1+x|$ explicitly, not the indefinite integral)

$$\ln|1+0| = \sum_{n=0}^{\infty}\frac{(-1)^n 0^{n+1}}{n+1} + C \implies 0 = C$$

So, $\ln|1+x| = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$, $R = 1$. What about the interval of convergence? The open interval is $(-1, 1)$, but since we integrated we need to check the endpoints.

<u>Check Endpoints</u>

At $x = 1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ converges by AST.

Note that this shows

$$\boxed{\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

At $x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} -\frac{1}{n+1}$ diverges (Harmonic Series). So $I = (-1, 1]$.

EXAMPLE 6.3.5

Find a power series for $f(x) = \dfrac{1}{(1-x)^3}$ about $x = 0$.

Solution. We know $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$ ($R = 1$).

So differentiate: $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ ($R = 1$).

Do it again: $\dfrac{2}{(1-x)^3} = \displaystyle\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ ($R = 1$).

Then we get $\dfrac{1}{(1-x)^3} = \dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty} n(n-1)x^{n-2}$ with $R = 1$.

<u>Check Endpoints</u>

At $x = 1$: $\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} n(n-1)$ diverges by the Divergence Test.

At $x = -1$: $\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} n(n-1)(-1)^{n-2}$ diverges by the Divergence Test.

So, $I = (-1, 1)$.

EXAMPLE 6.3.6

Find a power series for $f(x) = \arctan(x)$ about $x = 0$.

Solution. We will first find a series for $\dfrac{1}{1+x^2}$, then integrate!

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2| < 1 \implies |x| < 1$ ($R = 1$).

So $\arctan(x) = \displaystyle\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{2n+1}\right] + C$.

Sub in $x = 0$ to get C: $\arctan(0) = 0 + C \implies C = 0$.

So $\arctan(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $R = 1$.

<u>Check Endpoints</u>

At $x = -1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges by AST.

At $x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by AST.

So $I = [-1, 1]$.

EXAMPLE 6.3.7

Evaluate $\int \frac{1}{2-x^5}\,dx$ as a power series about $x=0$.

Solution. First, find a series for $\frac{1}{2-x^5}$:

$$\frac{1}{2-x^5} = \frac{1}{2}\left(\frac{1}{1-x^5/2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^5}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{5n}}{2^{n+1}}$$

for $|x^5/2| < 1 \implies |x| < 2^{1/5}$ ($R = 2^{1/5}$).
Then integrate:

$$\int \frac{1}{2-x^5}\,dx = \int \sum_{n=0}^{\infty}\frac{x^{5n}}{2^{n+1}}\,dx = \sum_{n=0}^{\infty}\left[\frac{x^{5n+1}}{2^{n+1}(5n+1)}\right] + C$$

with $R = 2^{1/5}$. We won't find C since we are evaluating an indefinite integral! The open interval is $(-2^{1/5}, 2^{1/5})$.

Check Endpoints

At $x = 2^{1/5}$: $\sum_{n=0}^{\infty}\frac{(2^{1/5})^{5n+1}}{2^{n+1}(5n+1)} = \sum_{n=0}^{\infty}\frac{2^n 2^{1/5}}{2^{n+1}(5n+1)} = \sum_{n=0}^{\infty}\left[\left(\frac{2^{1/5}}{2}\right)\left(\frac{1}{5n+1}\right)\right]$ diverges, use LCT with $\sum_{n=1}^{\infty}\frac{1}{n}$ (exercise).

At $x = -2^{1/5}$: $\sum_{n=0}^{\infty}\left[(-1)^{5n+1}\left(\frac{2^{1/5}}{2}\right)\left(\frac{1}{5n+1}\right)\right]$ converges by AST.

So $I = [-2^{1/5}, 2^{1/5})$.

Using differentiation, we can find another series for e^x.

PROPOSITION 6.3.8

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ *for all* $x \in \mathbb{R}$.

Proof of 6.3.8

We know $R = \infty$ for that series. Let $g(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. Then,

$$g'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = g(x)$$

So $g'(x) = g(x)$.
Solve this ODE, and we get $g(x) = Ce^x$, but by definition $g(0) = 1$, so $C = 1$ and therefore $g(x) = e^x$.

We will come back and explore this and other functions soon!

6.5 Review of Taylor Polynomials

DEFINITION 6.5.1: n-th degree Taylor polynomial

If f is n-times differentiable at $x = a$, the **nth degree Taylor polynomial** for f centred at $x = a$ is

$$T_{n,a}(x) = \sum_{n=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

DEFINITION 6.5.2: n-th degree Taylor remainder function

If f is n-times differentiable at $x = a$, we define the **nth degree Taylor remainder function** centred at $x = a$ to be:

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

The **error** in using $T_{n,a}(x)$ to approximate $f(x)$ is given by Error $= |R_{n,a}(x)|$.

To estimate the size of the error we use:

THEOREM 6.5.3: Taylor's Theorem

Assume f is $n+1$-times differentiable on an interval I containing $x = a$. Let $x \in I$. Then, there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

COROLLARY 6.5.4: Taylor's Inequality

$$|R_{n,a}(x)| \leqslant \frac{M|x-a|^{n+1}}{(n+1)!}$$

where $|f^{(n+1)}(c)| \leqslant M$ for all c between x and a.

6.7 Taylor Series and Convergence

Last week we examined how to obtain power series representations for certain functions, e.g., functions related to $\dfrac{1}{1-x}$, but is there a more general method? Let's see!

Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$ for $|x-a| < R$, $R > 0$. What are the a_n's?

First, at $x = a$: $f(a) = a_0$. Next: differentiate!

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

at $x = a$: $f'(a) = a_1$. Keep going!

$$f''(x) = 2a_2 + 6a_3(x-a) + \cdots \implies f''(a) = 2a_2$$

Therefore $a_2 = \dfrac{f''(a)}{2}$.

Another iteration gives $a_3 = \dfrac{f^{(3)}(a)}{6} = \dfrac{f^{(3)}(a)}{3!}$, etc.

In general: $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Hey look! We just proved:

THEOREM 6.7.1

If $f(x)$ has a power series representation about $x = a$, say $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for $|x-a| < R$, $R > 0$, then

$$a_n = \frac{f^{(n)}(a)}{n!}$$

That is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Called the ***Taylor series*** *for f centred at $x = a$.*

Special case: $a = 0$: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is called the **Maclaurin series** for f.

REMARK 6.7.2

The theorem has some strengths and weaknesses:

- Strength: The theorem says that *no matter how* you find a series expansion for a function:
  - Manipulating known series
  - Integrating/differentiating
  - Using the Taylor series formula

  you will get the Taylor series for f.
- Weakness: The theorem *assumes* f has a power series expansion, and concludes it must be the Taylor series. It doesn't say that *every* function is equal to its Taylor series.

Indeed, some functions are *not* equal to their Taylor series. For example,

$$f(x) = \begin{cases} 1/e & \text{if } x < -1 \\ e^x & \text{if } -1 \leqslant x \leqslant 1 \\ e & \text{if } x > 1 \end{cases}$$

Let's find f's Maclaurin series! Clearly $f^{(n)}(0) = 1$ for all n, so f's Maclaurin series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which we all know converges on $\mathbb{R}$. But we also know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$, but that means $f(2) = e \neq e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}$.

So, while both f and the series exist everywhere, $f(x) \neq \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

This means we need to develop a way to determine if a function, f, is equal to its Taylor Series on the interval of convergence.

The first thing we need to notice is that the partial sums of a Taylor series are the Taylor polynomials!

So, what we want to determine is for which $x \in \mathbb{R}$ is

$$f(x) = \lim_{n\to\infty} T_{n,a}(x)$$

Or, since we know $f(x) = T_{n,a}(x) + R_{n,a}(x)$, we need to check if

$$\lim_{n\to\infty} R_{n,a}(x) = 0$$

For each x where $R_{n,a}(x) \to 0$, we can conclude that $f(x)$ is equal to its Taylor series.

In order to show that $R_{n,a}(x) \to 0$, it would be great to have a way to approximate its size, and we do!

THEOREM 6.7.3: Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \leqslant M$ *for* $|x-a| \leqslant d \in \mathbb{R}$, *then*

$$|R_{n,a}(x)| \leqslant \frac{M|x-a|^{n+1}}{(n+1)!}$$

for $|x-a| < d$.

With this, we establish the convergence theorem!

THEOREM 6.7.4: Convergence Theorem for Taylor Series

Assume f *has derivatives of all orders on an interval* I *containing* $x = a$.
Assume also that there exists $M \in \mathbb{R}$ *such that* $|f^{(k)}(x)| \leqslant M$ *for all* $k \in \mathbb{N}$ *and* $x \in I$. *Then,*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for $x \in I$.

Proof of 6.7.4

First, if $x = a$ then $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(a-a)^n = f(a)$, so we only need to prove it for $x \neq a$. Say $x_0 \in I$, $x_0 \neq a$. Taylor's Inequality says that since $|f^{(n+1)}(x)| \leqslant M$,

$$0 \leqslant |R_{n,a}(x_0)| \leqslant \frac{M|x_0-a|^{n+1}}{(n+1)!}$$

Also, we have already shown that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all $x \in \mathbb{R}$, so $\displaystyle\lim_{n\to\infty} \frac{M|x_0-a|^{n+1}}{(n+1)!} = 0$. Therefore, by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} |R_{n,a}(x_0)| = 0$, as desired.

COROLLARY 6.7.5

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *for all* $x \in \mathbb{R}$.

Proof of 6.7.5

Fix $B > 0$, then for $f(x) = e^x$, $|f^{(n+1)}(x)| = e^x \leqslant e^B$ for all $x \in [-B, B]$. Therefore, by the Convergence Theorem, $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in [-B, B]$. But B was arbitrary, so $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

COROLLARY 6.7.6

Both $\sin(x)$ *and* $\cos(x)$ *are equal to their Maclaurin series for all* $x \in \mathbb{R}$.

Proof of 6.7.6

Both functions are infinitely differentiable on $\mathbb{R}$, and their derivatives (namely $\pm\sin(x)$ or $\pm\cos(x)$) are bounded above by 1. So, by the Convergence Theorem, the result follows.

Now we know that $\sin(x)$ and $\cos(x)$ are equal to their Maclaurin series for all $x \in \mathbb{R}$, but we haven't determined what they are!

Let's start with the Maclaurin series for $\cos(x)$.

- $f(x) = \cos(x) \implies f(0) = 1$
- $f'(x) = -\sin(x) \implies f'(0) = 0$
- $f''(x) = -\cos(x) \implies f''(0) = -1$
- $f^{(3)}(0) = \sin(x) \implies f^{(3)}(0) = 0$
- $f^{(4)}(0) = \cos(x) \implies f^{(4)}(0) = 1$

this repeats.

So, the series is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

So, for all $x \in \mathbb{R}$,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

What about $\sin(x)$? We could use the formula, or integrate the series for $\cos(x)$.

$$\sin(x) = \int \cos(x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n)!(2n+1)}\right] + C$$

But $\sin(0) = 0 \implies 0 = 0 + C \implies C = 0$. So, for all $x \in \mathbb{R}$,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

The point is: we can use whatever methods we want to find a Taylor/Maclaurin series, we don't always need to use the formula!

EXAMPLE 6.7.7

Find the Taylor series for e^x centred at $x = 3$.

Solution. We want powers of $(x-3)$, and we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So,

$$e^x = e^{x-3+3} = e^3 e^{x-3} = e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

EXAMPLE 6.7.8

Find the Maclaurin Series for $f(x) = x^2 \sin(x)$.
Solution.

$$x^2 \sin(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

for all $x \in \mathbb{R}$ with $R = \infty$.

EXAMPLE 6.7.9

Find the Taylor series about $x = \frac{\pi}{2}$ for $f(x) = \sin(x)$.
Solution.

$$\begin{aligned}
\sin(x) &= \sin\left[\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right] \\
&= \sin\left(x - \frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(x - \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) && \text{trig. identity} \\
&= \cos\left(x - \frac{\pi}{2}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}
\end{aligned}$$

for all $x \in \mathbb{R}$ with $R = \infty$.

6.9 Binomial Series

Let's fine one more series: Binomial series!

We know the Binomial Theorem for $(1+x)^k$ where $n \in \mathbb{N}$:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The question is: can we extend this to $(1+x)^n$ for any $n \in \mathbb{R}$? Yes! We can find its Maclaurin series!

- $f(x) = (1+x)^n \implies f(0) = 1$
- $f'(x) = n(1+x)^{n-1} \implies f'(0) = n$
- $f''(n) = n(n-1)(1+x)^{n-2} \implies f''(0) = n(n-1)$
- $\vdots$
- $f^{(k)}(x) = n(n-1)\cdots[n-(k-1)]\,(1+x)^{n-k} \implies f^{(k)}(0) = n(n-1)\cdots(n-k+1)$

So, we get

$$\sum_{k=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k$$

for the Maclaurin series.

First, let's determine the radius of convergence, for $n \neq 0, 1, 2, \ldots$: Ratio Test:

$$\lim_{k\to\infty} \left| \left(\frac{n(n-1)\cdots(n-k+1)(n-k)x^{k+1}}{(k+1)!} \right) \left(\frac{k!}{n(n-1)\cdots(n-k+1)x^k} \right) \right| = \lim_{k\to\infty} \left| \frac{n-k}{k+1} \right| |x| = |x|$$

Need $|x| < 1$, so $R = 1$, and the open interval is $(-1, 1)$.

What about endpoint convergence? Here is the answer, but you won't be expected to know this:

Interval of convergence:

- $[-1,1]$ if $n>0$, $n\notin\mathbb{N}$
- $(-1,1]$ if $-1<n<0$
- $(-1,-1)$ if $n\leqslant -1$
- $\mathbb{R}$ if $n=0,1,2,\cdots$

Notation:

$$\binom{n}{k}=\frac{n(n-1)\cdots(n-k+1)}{k!}$$

called the **Binomial coefficients** with k-terms in the numerator. Keep in mind that $\binom{n}{0}=1$.

The bigger question is: does $(1+x)^n$ equal to its Maclaurin series on $(-1,1)$? The answer is yes! Let's prove it. We could try to use the Convergence Theorem, but instead we will prove it directly!

First, we claim that

$$\binom{n}{k+1}(k+1)+\binom{n}{k}k=\binom{n}{k}n$$

for $k\geqslant 1$.

Proof:

$$\begin{aligned}
\binom{n}{k+1}(k+1)+\binom{n}{k}k &= \frac{n(n-1)\cdots(n-k+1)(n-k)}{(k+1)!}(k+1)+\frac{n(n-1)\cdots(n-k+1)}{k!}(k)\\
&= \frac{n(n-1)\cdots(n-k+1)}{n!}(n-k)+\frac{n(n-1)\cdots(n-k+1)}{k!}(k)\\
&= \binom{n}{k}(n-k)+\binom{n}{k}k\\
&= \binom{n}{k}(n-k+k)\\
&= \binom{n}{k}n
\end{aligned}$$

Next, let $f(x)=\sum\limits_{k=0}^{\infty}\binom{n}{k}x^k$. We claim that

$$f'(x)+xf'(x)=nf(x)$$

for all $x\in(-1,1)$.

Proof:

$$\begin{aligned}
f'(x) + xf'(x) &= \sum_{k=1}^{\infty} \binom{n}{k} k x^{k-1} + \sum_{k=1}^{\infty} \binom{n}{k} k x^k \\
&= \binom{n}{1} + \sum_{k=2}^{\infty} \binom{n}{k} k x^{k-1} + \sum_{k=1}^{\infty} \binom{n}{k} k x^k \\
&= \binom{n}{1} + \sum_{k=1}^{\infty} \binom{n}{k+1} (k+1) x^k + \sum_{k=1}^{\infty} \binom{n}{k} k x^k \\
&= \binom{n}{1} + \sum_{k=1}^{\infty} \left[\binom{n}{k+1} (k+1) + \binom{n}{k} k \right] x^n \\
&= \binom{n}{1} + \sum_{k=1}^{\infty} \binom{n}{k} n x^k \\
&= n + \sum_{k=1}^{\infty} \binom{n}{k} n x^k \\
&= n \left[1 + \sum_{k=1}^{\infty} \binom{n}{k} x^k \right] \\
&= n \sum_{k=0}^{\infty} \binom{n}{k} x^k \\
&= nf(x)
\end{aligned}$$

Finally, let $g(x) = \dfrac{f(x)}{(1+x)^n}$. Let's show $g'(x) = 0$ for $x \in (-1,1)$:

$$\begin{aligned}
g'(x) &= \frac{f'(x)(1+x)^n - f(x)n(1+x)^{n-1}}{(1+x)^{2n}} \\
&= \frac{f'(x)(1+x)^n - (1+x)f'(x)(1+x)^{n-1}}{(1+x)^{2n}} && \text{by previous claim} \\
&= \frac{f'(x)(1+x)^n - f'(x)(1+x)^n}{(1+x)^{2n}} \\
&= 0
\end{aligned}$$

So, $g'(x) = 0$ for all $x \in (-1,1)$, which means g is constant on $(-1,1)$.

Since $f(0) = 1$, we get $g(0) = 1/1 = 1$, so $g(x) = 1$ for all $x \in (-1,1)$. This implies $f(x) = (1+x)^n$ for $x \in (-1,1)$. We have finally proven:

THEOREM 6.9.1: Generalized Binomial Theorem

Let $n \in \mathbb{R}$, then for all $x \in (-1,1)$:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

and $\binom{n}{0} = 1$.

EXAMPLE 6.9.2

Find the Maclaurin series for $\arcsin(x)$.
Solution.
Step 1: Find the Maclaurin series for $(1+x)^{-1/2}$.

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-k+1)}{k!}x^k = \sum_{k=0}^{\infty} \frac{(-1)^k(1)(3)(5)\cdots(2k-1)}{2^k(k!)}x^k$$

for $x \in (-1,1)$.
Step 2: Find the Maclaurin series for $(1-x^2)^{-1/2}$.

$$(1-x^2)^{-1/2} = \left[1+(-x^2)\right]^{-1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k(1)(3)(5)\cdots(2k-1)}{2^k(k!)}(-x^2)^k = \sum_{k=0}^{\infty} \frac{(1)(3)(5)\cdots(2k-1)}{2^k}x^{2k}$$

for $|-x^2| < 1 \implies |x| < 1$ with $x \in (-1,1)$.
Step 3: Integrate!

$$\arcsin(x) = \sum_{k=0}^{\infty} \frac{(1)(3)(5)\cdots(2k-1)x^{2k+1}}{2^k(k!)(2k+1)}$$

for $x \in (-1,1)$ with $C=0$ since $\arcsin(0)=0$.

6.10 Additional Examples and Applications of Taylor Series

The applications we will examine are:

(I) Finding sums

(II) Evaluating limits

(III) Evaluating and approximating integrals

Recap of Known Series:

- $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ (R=0)$
- $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ (R=\infty)$
- $\displaystyle \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \ (R=\infty)$
- $\displaystyle \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \ (R=\infty)$
- $\displaystyle (1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!}x^k \ (R=1)\ (R=\infty)$

Finding Sums

Given a series, we may be able to manipulate it into one of the above series and find the sum that way. Alternatively, we could manipulate a known series into the given series!

EXAMPLE 6.10.1

(i) Find the sum of $\sum_{n=0}^{\infty}\left(\frac{n+1}{n!}\right)x^n = S(x)$.

Solution. This is almost e^x, but it has an extra "$n+1$" Let's integrate!

$$\int S(x)\,dx = \sum_{n=0}^{\infty}\left(\frac{x^{n+1}}{n!}\right) + C = x\sum_{n=0}^{\infty}\frac{x^n}{n!} + C = xe^x + C$$

So $S(x) = (x + e^x + C)' = e^x + xe^x$.

(ii) $\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{2n+1}\right] + 4 = S_2(x)$.

Solution. Differentiate:

$$S_2'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1+x^2}$$

so $S_2(x) = \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$. But $S_2(0) = 4$, so $C = 4$. Thus, $S_2(x) = \arctan(x) + 4$.

(iii) $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!} = S_3(x)$

Solution.

$$S_3(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\,(\pi/2)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{2}\right) = 0$$

(iv) Starting with $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n$, find $\sum_{n=1}^{\infty}\frac{nx^n}{7}$.

Solution.

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1}$$
$$\implies \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty}nx^n$$
$$\implies \frac{x}{7(1-x)^2} = \sum_{n=1}^{\infty}\frac{nx^n}{7}$$

(v) $S_5(x) = \sum_{n=0}^{\infty}\frac{e(e-1)\cdots(e-n+1)}{3^n(n!)}$

Solution.

$$S_5(x) = \sum_{n=0}^{\infty}\frac{e(e-1)\cdots(e-n+1)}{n!}\left(\frac{1}{3}\right)^n = \left(1+\frac{1}{3}\right)^e = \left(\frac{4}{3}\right)^e$$

Evaluating Limits

We can use Taylor series to evaluate limits, instead of L'Hopital's Rule. This idea is similar to how we used Taylor polynomials and Taylor's Approximation Theorem I to evaluate limits in MATH 137.

EXAMPLE 6.10.2

Evaluate with series and not L'Hopital's Rule.

(i) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$.

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{e^x-1}{x} &= \lim_{x\to 0} \frac{(1+x+x^2/2!+\cdots)-1}{x}\\ &= \lim_{x\to 0}\frac{x+x^2/2!+\cdots}{x}\\ &= \lim_{x\to 0}\left[1+x/2!+\cdots\right]\\ &= 1\end{aligned}$$

(ii) $\lim\limits_{x\to 0} \dfrac{1-\cos(x)}{x^2}$.

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(x)}{x^2} &= \lim_{x\to 0} \frac{1-(1-x^2/2!+x^4/4!-\cdots)}{x^2}\\ &= \lim_{x\to 0}\frac{x^2/2!+x^4/4!}{x^2}\\ &+ \lim_{x\to 0}\left[1/2!-x^2/4!+\cdots\right]\\ &= \frac{1}{2}\end{aligned}$$

(iii) $\lim\limits_{x\to 0} \dfrac{e^x - x^2/2 - x - 1}{\sin(x) - x}$.

Solution.

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - x^2/2 - x - 1}{\sin(x)-x} &= \lim_{x\to 0} \frac{(1+x+x^2/2!+x^3/3!+\cdots)-x^2/2-x-1}{(x-x^3/3!+x^5/5!-\cdots)-x}\\ &= \lim_{x\to 0}\frac{x^3/3!+x^4/4!+\cdots}{-x^3/3!+x^5/5!-\cdots}\\ &= \lim_{x\to 0}\frac{1/3!+x/4!+\cdots}{-1/3!+x^2/5!-\cdots}\\ &= \frac{1/3!}{-1/3!}\\ &= -1\end{aligned}$$

Evaluating Integrals as Series

EXAMPLE 6.10.3

Evaluate $\int e^{-x^2}\,dx$ as a series.

Solution.

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\right] + C$$

EXAMPLE 6.10.4

How many terms would we need to use to approximate $\int_0^1 e^{-x^2}\,dx$ to an accuracy of $\dfrac{1}{10!(21)}$?

Solution.

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{n!(2n+1)}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}$$

This converges by AST! Let's use AST estimation. First, write out some terms:

$$1-\frac{1}{1!(3)}+\frac{1}{2!(5)}-\frac{1}{3!(7)}+\frac{1}{4!(9)}-\frac{1}{5!(11)}+\frac{1}{6!(13)}-\frac{1}{7!(15)}+\frac{1}{8!(17)}-\frac{1}{9!(19)}+\underbrace{\boxed{\frac{1}{10!(21)}}}_{\text{error}}$$

So, the estimate needs at least 10 terms.

Recap of Power Series

Strategy for Solving Questions:

- Given a *series*, to find radius and interval of convergence:
  - Ratio test for R and open interval
  - Check endpoints with other tests
- Given a *series*, to find its sum:
  - Relate it to a known Series
  - May need to integrate/differentiate
- Given a *function*, to get its Taylor/Maclaurin series, we can:
  - Use the Taylor series formula $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ where R and I need to be found from scratch in this case.
  - Manipulate/Integrate/Differentiate a known series. R will be known, but endpoints need to be checked to find I.
  - If asked for a Taylor series about $x=a$, try $f(x)=f(x-a+a)$, manipulate, and use a known series.
- Stuff we can do with Taylor series:
  (I) Find sums
  (II) Evaluate limits
  (III) Evaluate and approximate integrals